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1

The Second-Order Self-Adjoint Equation with Mixed Derivatives

Kirsten Messer

1 Introduction

In this chapter, we are concerned with the second-order, self-adjoint dynamic equation $(p(t)x^\Delta)^\nabla + q(t)x = 0$ on a time scale. When $\mathbb{T} = \mathbb{R}$, this reduces to the usual self-adjoint differential equation, $(p(t)x')' + q(t)x = 0$.

This equation contains both the Δ and ∇ derivatives, and so we begin in Section 2 by establishing several results concerning the interaction of these two derivatives. Also included in this section is a theorem which shows that under certain conditions, the generalized exponential functions $e_p(t, t_0)$ and $\hat{e}_r(t, t_0)$ can be related to one another. In Section 3, we examine three second-order linear dynamic equations and demonstrate that they can be written in self-adjoint form. Additionally, we present solution techniques for one of the three equations. The first results which are directly related to the self-adjoint equation are contained in Section 4, which culminates with a reduction of order theorem. We turn our attention to oscillation and disconjugacy in Section 5, where we establish an analogue of the Sturm Separation Theorem, and, via the Polya and Trench factorizations demonstrate the existence of recessive and dominant solutions of the self-adjoint equation. The final section of the chapter, Section 6, discusses Riccati techniques as they relate to the self-adjoint equation. The material in this chapter has been previously published in [92] and [93].

2 Preliminary Results

One of the fundamental tools which is used in the study of differential and difference equations is L'Hôpital's Rule. A version of this crucial theorem for Δ-derivatives, Theorem 1.3, can be found in [31, Theorem 1.119]. It is presented here in a slightly different form. A similar result, Theorem 1.4, was developed in [92] for ∇-derivatives.

We may want to employ L'Hôpital's Rule to evaluate a limit as $t \to \pm\infty$, so we make the following definitions.

Definition 1.1. Let $\varepsilon > 0$. If $\mathbb{T}$ is unbounded above, we define a *left neighborhood of* ∞, which we denote by $L_\varepsilon(\infty)$, by

$$L_\varepsilon(\infty) = \left\{ t \in \mathbb{T} : t > \frac{1}{\varepsilon} \right\}.$$

Similarly, if $\mathbb{T}$ is unbounded below, we define a *right neighborhood of* $-\infty$, denoted $R_\varepsilon(-\infty)$ by

$$R_\varepsilon(-\infty) = \left\{ t \in \mathbb{T} : t < -\frac{1}{\varepsilon} \right\}.$$

We next define right and left neighborhoods for points in $\mathbb{T}$.

Definition 1.2. Let $\varepsilon > 0$. For any right-dense $t_0 \in \mathbb{T}$, define a *right neighborhood of* t_0, denoted $R_\varepsilon(t_0)$, by

$$R_\varepsilon(t_0) := \{t \in \mathbb{T} : 0 < t - t_0 < \varepsilon\}.$$

Similarly, for any left-dense $t_0 \in \mathbb{T}$, define a *left neighborhood of* t_0, denoted $L_\varepsilon(t, t_0)$, by

$$L_\varepsilon(t_0) := \{t \in \mathbb{T} : 0 < t_0 - t < \varepsilon\}.$$

Theorem 1.3 (L'Hôpital's Rule for Δ-derivatives). *Assume f and g are Δ-differentiable on $\mathbb{T}$, and let $t_0 \in \mathbb{T} \cup \{\infty\}$. If $t_0 \in \mathbb{T}$, assume t_0 is left-dense. Furthermore, assume*

$$\lim_{t \to t_0^-} f(t) = \lim_{t \to t_0^-} g(t) = 0,$$

and suppose there exists $\varepsilon > 0$ with

$$g(t)g^\Delta(t) < 0 \quad \text{for all} \quad t \in L_\varepsilon(t_0).$$

Then we have

$$\liminf_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} \leq \liminf_{t \to t_0^-} \frac{f(t)}{g(t)} \leq \limsup_{t \to t_0^-} \frac{f(t)}{g(t)} \leq \limsup_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)}.$$

Theorem 1.4 (L'Hôpital's Rule for ∇-derivatives.). *Assume f and g are ∇-differentiable on $\mathbb{T}$ and let $t_0 \in \mathbb{T} \cup \{-\infty\}$. If $t_0 \in \mathbb{T}$, assume t_0 is right-dense. Furthermore, assume*

$$\lim_{t \to t_0^+} f(t) = \lim_{t \to t_0^+} g(t) = 0,$$

and suppose there exists $\varepsilon > 0$ with

$$g(t)g^\nabla(t) > 0 \quad \text{for all} \quad t \in R_\varepsilon(t_0).$$

Then

$$\liminf_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)} \leq \liminf_{t \to t_0^+} \frac{f(t)}{g(t)} \leq \limsup_{t \to t_0^+} \frac{f(t)}{g(t)} \leq \limsup_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)}.$$

Proof. The proofs of these theorems are very similar, and we only include the proof of the ∇-derivative statement here. Without loss of generality, assume $g(t)$ and $g^{\nabla}(t)$ are both strictly positive on $R_{\varepsilon}(t_0)$.

Let $\delta \in (0, \varepsilon]$, and let $a := \inf_{\tau \in R_{\delta}(t_0)} \frac{f^{\nabla}(\tau)}{g^{\nabla}(\tau)}$, $b := \sup_{\tau \in R_{\delta}(t_0)} \frac{f^{\nabla}(\tau)}{g^{\nabla}(\tau)}$. To complete the proof, it suffices to show

$$a \leq \inf_{\tau \in R_{\delta}(t_0)} \frac{f(\tau)}{g(\tau)} \leq \sup_{\tau \in R_{\delta}(t_0)} \frac{f(\tau)}{g(\tau)} \leq b,$$

as we may then let $\delta \to 0$ to obtain the desired result.

We must be careful here, as either a or b could possibly be infinite. Note, however, that since $g^{\nabla}(\tau) > 0$ on $R_{\delta}(t_0)$, we have $a < \infty$. Similarly, $b > -\infty$. So our only concern is if $a = -\infty$ or $b = \infty$. But, if $a = -\infty$, we have immediately that

$$a \leq \inf_{\tau \in R_{\delta}(t_0)} \frac{f(\tau)}{g(\tau)},$$

as desired, and if $b = \infty$, we have immediately that

$$\sup_{\tau \in R_{\delta}(t_0)} \frac{f(\tau)}{g(\tau)} \leq b,$$

as desired. Therefore, we may assume that both a and b are finite. Then

$$ag^{\nabla}(\tau) \leq f^{\nabla}(\tau) \leq bg^{\nabla}(\tau) \quad \text{for all} \quad \tau \in R_{\delta}(t_0),$$

and by a theorem of Guseinov and Kaymakçalan [64, Theorem 3.4],

$$\int_t^s ag^{\nabla}(\tau) \, \nabla\tau \leq \int_t^s f^{\nabla}(\tau) \, \nabla\tau \leq \int_t^s bg^{\nabla}(\tau) \, \nabla\tau \quad \text{for all} \quad s, t \in R_{\delta}(t_0), \ t < s.$$

Integrating, we see that

$$ag(s) - ag(t) \leq f(s) - f(t) \leq bg(s) - bg(t) \quad \text{for all} \quad s, t \in R_{\delta}(t_0), \ t < s.$$

Letting $t \to t_0^+$, we get

$$ag(s) \leq f(s) \leq bg(s) \quad \text{for all} \quad s \in R_{\delta}(t_0),$$

and thus

$$a \leq \inf_{s \in R_{\delta}(t_0)} \frac{f(s)}{g(s)} \leq \sup_{s \in R_{\delta}(t_0)} \frac{f(s)}{g(s)} \leq b.$$

Then, by the discussion above, the proof is complete. $\square$

Remark 1.5. Although these theorems are only stated in terms of one-sided limits, analogous results can be established if the limit is taken from the other direction. To apply L'Hôpital's rule using Δ-derivatives and a right-sided limit, t_0 must be right-dense (or $-\infty$ if $\mathbb{T}$ is unbounded below), and gg^{Δ} must be

strictly positive on a right neighborhood of t_0. Similarly, to apply L'Hôpital's rule using ∇-derivatives and a left-sided limit, t_0 must be left-dense (or ∞ if $\mathbb{T}$ is unbounded above), and gg^∇ must be strictly negative on some left neighborhood of t_0.

Recall that our dynamic equation contains both Δ and ∇ derivatives, and we want to determine how the two types of derivatives interact with one another. The interaction of these derivatives is closely tied to the function compositions $\sigma(\rho(t))$ and $\rho(\sigma(t))$. Since $\sigma(\rho(t)) \neq t$ at points which are left-dense and right-scattered, we need to consider these points separately in some instances. Similarly, $\rho(\sigma(t)) \neq t$ at points which are right-dense and left-scattered, so we will occasionally need to consider these points separately as well. To simplify the notation, we define the following sets. Let

$$A := \{t \in \mathbb{T} \mid t \text{ is left-dense and right-scattered}\}, \quad \text{and} \quad \mathbb{T}_A := \mathbb{T} \setminus A.$$

Similarly, let

$$B := \{t \in \mathbb{T} \mid t \text{ is right-dense and left-scattered}\}, \quad \text{and} \quad \mathbb{T}_B := \mathbb{T} \setminus B.$$

Lemma 1.6. *If $t \in \mathbb{T}_A$, then $\sigma(\rho(t)) = t$. If $t \in \mathbb{T}_B$, then $\rho(\sigma(t)) = t$.*

Proof. We will only prove the first statement. The proof of the second statement is similar (see Exercise 1.7). If $t \in \mathbb{T}_A$, then either t is left-scattered, or t is both left-dense and right-dense. If t is left-scattered, then $\rho(t)$ is right-scattered and it is clear that $\sigma(\rho(t)) = t$. If t is both left-dense and right-dense, then $\sigma(t) = t$ and $\rho(t) = t$. Hence $\sigma(\rho(t)) = \sigma(t) = t$. In either case we get the desired result. $\square$

Exercise 1.7. Prove the second statement of Lemma 1.6.

The following theorem allows us to interchange the Δ and ∇ derivatives.

Theorem 1.8. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and f^Δ is rd-continuous on $\mathbb{T}^\kappa$, then f is ∇-differentiable on $\mathbb{T}_\kappa$, and*

$$f^\nabla(t) = \begin{cases} f^\Delta(\rho(t)) & t \in \mathbb{T}_A \\[2mm] \lim_{s \to t^-} f^\Delta(s) & t \in A. \end{cases}$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_\kappa$ and g^∇ is ld-continuous on $\mathbb{T}_\kappa$, then g is Δ-differentiable on $\mathbb{T}^\kappa$, and

$$g^\Delta(t) = \begin{cases} g^\nabla(\sigma(t)) & t \in \mathbb{T}_B \\[2mm] \lim_{s \to t^+} g^\nabla(s) & t \in B. \end{cases}$$

Proof. We will only prove the first statement. The proof of the second statement is similar (see Exercise 1.9). First, assume $t \in \mathbb{T}_A$. Then there are two cases: Either

(i) t is left-scattered, or

(ii) t is both left-dense and right-dense.

Case (i): Suppose t is left-scattered and f is Δ-differentiable on $\mathbb{T}^\kappa$. Then f is continuous at t, and is therefore ∇-differentiable at t. Next, note that $\rho(t)$ is right-scattered, and

$$
\begin{aligned}
f^\Delta(\rho(t)) &= \frac{f(\sigma(\rho(t))) - f(\rho(t))}{\sigma(\rho(t)) - \rho(t)} \\
&= \frac{f(t) - f(\rho(t))}{t - \rho(t)} \\
&= f^\nabla(t).
\end{aligned}
$$

Case (ii): Now, suppose t is both left-dense and right-dense, and $f : \mathbb{T} \to \mathbb{R}$ is continuous on $\mathbb{T}$ and Δ-differentiable at t. Since t is right-dense and f is Δ-differentiable at t, we have that

$$
\lim_{s \to t} \frac{f(t) - f(s)}{t - s}
$$

exists. But t is left-dense as well, so this expression also defines $f^\nabla(t)$, and we see that

$$
\begin{aligned}
f^\nabla(t) &= \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \\
&= f^\Delta(t) \\
&= f^\Delta(\rho(t)).
\end{aligned}
$$

So, we have established the desired result in the case where $t \in \mathbb{T}_A$.

Now suppose $t \in A$. Then t is left-dense. Hence $f^\nabla(t)$ exists provided

$$
\lim_{s \to t} \frac{f(t) - f(s)}{t - s}
$$

exists.

As t is right-scattered, we need only consider the limit as $s \to t$ from the left. Then we apply L'Hôpital's rule [31, Theorem 1.119], differentiating with respect to s to get

$$
\lim_{s \to t-} \frac{f(t) - f(s)}{t - s} = \lim_{s \to t-} \frac{-f^\Delta(s)}{-1} = \lim_{s \to t-} f^\Delta(s).
$$

Since we have assumed that f^Δ is rd-continuous, this limit exists. Hence f is ∇-differentiable, and $f^\nabla(t) = \lim_{s \to t-} f^\Delta(t)$, as desired. $\qquad\square$

Exercise 1.9. Prove the second statement of Theorem 1.8.

Corollary 1.10. *If $t_0 \in \mathbb{T}$, and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$, then $\int_{t_0}^{t} f(\tau) \Delta \tau$ is ∇-differentiable on $\mathbb{T}$ and*

$$\left[\int_{t_0}^{t} f(\tau) \Delta \tau \right]^{\nabla} = \begin{cases} f(\rho(t)) & t \in \mathbb{T}_A \\ \lim_{s \to t^-} f(s) & t \in A. \end{cases}$$

If $t_0 \in \mathbb{T}$, and $g : \mathbb{T} \to \mathbb{R}$ is ld-continuous on $\mathbb{T}$, then $\int_{t_0}^{t} g(\tau) \nabla \tau$ is Δ-differentiable on $\mathbb{T}$ and

$$\left[\int_{t_0}^{t} g(\tau) \nabla \tau \right]^{\Delta} = \begin{cases} g(\sigma(t)) & t \in \mathbb{T}_B \\ \lim_{s \to t^+} g(s) & t \in B. \end{cases}$$

The following corollary was previously established by Atici and Guseinov in their work [20].

Corollary 1.11. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^{\kappa}$ and if f^{Δ} is continuous on $\mathbb{T}^{\kappa}$, then f is ∇-differentiable on $\mathbb{T}_{\kappa}$ and*

$$f^{\nabla}(t) = f^{\Delta \rho}(t) \quad \text{for } t \in \mathbb{T}_{\kappa}.$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}^{\kappa}$ and if g^{∇} is continuous on $\mathbb{T}_{\kappa}$, then g is Δ-differentiable on $\mathbb{T}^{\kappa}$ and

$$g^{\Delta}(t) = g^{\nabla \sigma}(t) \quad \text{for } t \in \mathbb{T}^{\kappa}.$$

Exercise 1.12. Let $a \in \mathbb{T}$ and calculate the following derivatives:

(i) $\left[\int_a^t \sigma(s) \, \Delta s \right]^{\nabla}$;

(ii) $\left[\int_a^t \rho(s) \, \nabla s \right]^{\Delta}$.

Now, there are a couple more integral formulas that will be useful, the first of which was established in [31, Theorem 1.75].

Lemma 1.13. *The following hold:*

(i) $\int_t^{\sigma(t)} f(s) \, \Delta s = \mu(t) f(t)$,

(ii) $\int_{\rho(t)}^{t} f(s) \, \nabla s = \nu(t) f(t)$,

(iii) $\int_t^{\sigma(t)} f(s) \, \nabla s = \mu(t) f^{\sigma}(t)$,

(iv) $\int_{\rho(t)}^{t} f(s) \, \Delta s = \nu(t) f^{\rho}(t)$.

Exercise 1.14. Prove Lemma 1.13.

The next material deals with the "generalized exponential functions", $e_p(t, t_0)$, and $\hat{e}_p(t, t_0)$, which were discussed in Chapters 1 and 3.

Lemma 1.15. *Let $p : \mathbb{T} \to \mathbb{R}$. Then p is regressive if and only if $-p^{\rho}$ is ν-regressive, and $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$ if and only if $1 + \nu(t)p^{\rho}(t) > 0$ for all $t \in \mathbb{T}$. Similarly, if $q : \mathbb{T} \to \mathbb{R}$, then q is ν-regressive if and only if $-q^{\sigma}$ is regressive, and $1 - \nu(t)q(t) > 0$ for all $t \in \mathbb{T}$ if and only if $1 - \mu(t)q^{\sigma}(t) > 0$ for all $t \in \mathbb{T}$.*

Proof. We will only prove the first statement. The proof of the second statement is similar (see Exercise 1.16). First, assume p is regressive. We then wish to show that $-p^{\rho}$ is ν-regressive, that is, we wish to show that $1 + \nu(t)(p^{\rho}(t)) \neq 0$.
Case 1: Fix $t \in \mathbb{T}_A$. Then $\rho(t) \in \mathbb{T}$, and as p is regressive, we have that

$$1 + \mu(\rho(t))p(\rho(t)) \neq 0,$$

so, using the definition of $\mu(t)$,

$$1 + [\sigma(\rho(t)) - \rho(t)]p(\rho(t)) \neq 0.$$

But $t \in \mathbb{T}_A$, so $\sigma(\rho(t)) = t$, and we get

$$1 + [t - \rho(t)]p^{\rho}(t) \neq 0,$$

or

$$1 + \nu(t)p^{\rho}(t) \neq 0,$$

as desired.
Case 2: Fix $t \in A$. Then t is left-dense and right-scattered, so $\nu(t) = 0$. Hence

$$1 + \nu(t)p^{\rho}(t) = 1 + 0p^{\rho}(t) = 1 \neq 0.$$

As $1 + \nu(t)p^{\rho}(t) \neq 0$ for any $t \in \mathbb{T}$, we see that $-p^{\rho}$ is ν-regressive.
Conversely, suppose $-p^{\rho}$ is ν-regressive. We then wish to show that p is regressive, that is, we wish to show that $1 + p(t)\mu(t) \neq 0$.
Case 1: Fix $t \in \mathbb{T}_B$. Then $\sigma(t) \in \mathbb{T}$, and, as $-p^{\rho}$ is ν-regressive, we have that

$$1 + \nu(\sigma(t))p^{\rho}(\sigma(t)) \neq 0,$$

so, using the definition of $\nu(t)$,

$$1 + [\sigma(t) - \rho(\sigma(t))]p(\rho(\sigma(t))) \neq 0.$$

But $t \in \mathbb{T}_B$, so $\rho(\sigma(t)) = t$, and we get

$$1 + [\sigma(t) - t]p(t) \neq 0,$$

or

$$1 + \mu(t)p(t) \neq 0,$$

as desired.

Case 2: Fix $t \in B$. Then t is right-dense and left-scattered, so $\mu(t) = 0$. Hence

$$1 + \mu(t)p(t) = 1 + 0p(t) = 1 \neq 0.$$

As $1 + \mu(t)p(t) \neq 0$ for any $t \in \mathbb{T}$, we see that p is regressive.

To show $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$ if and only if $1 + \nu(t)p^\rho(t) > 0$ for all $t \in \mathbb{T}$, simply replace "$\neq 0$" by "> 0" in the preceding proof. $\square$

Exercise 1.16. Prove the second statement of Theorem 1.15.

Theorem 1.17 (Equivalence of delta and nabla exponential functions).
If p is continuous and regressive, then

$$e_p(t, t_0) = \hat{e}_{\frac{p^\rho}{1+p^\rho\nu}}(t, t_0) = \hat{e}_{\ominus_\nu(-p^\rho)}(t, t_0).$$

If q is continuous and ν-regressive, then

$$\hat{e}_q(t, t_0) = e_{\frac{q^\sigma}{1-q^\sigma\mu}}(t, t_0) = e_{\ominus(-q^\sigma)}(t, t_0).$$

Proof. We will only prove the first statement. The proof of the second statement is similar (see Exercise 1.18). Suppose that $p : \mathbb{T} \to \mathbb{R}$ is continuous and regressive, then by Lemma 1.15 we have that $-p^\rho$ is ν-regressive. Furthermore, since p is continuous, $-p^\rho$ is ld-continuous. Hence $-p^\rho \in \mathcal{R}_\nu$. Then as $\mathcal{R}_\nu$ is an Abelian group under $\oplus_\nu$, we see that $\ominus_\nu(-p^\rho) = \frac{p^\rho}{1+p^\rho\nu} \in \mathcal{R}_\nu$, and therefore $\hat{e}_{\frac{p^\rho}{1+p^\rho\nu}}(t, t_0)$ exists.

To complete the proof, it therefore suffices to show that $e_p(t, t_0)$ solves the initial value problem

$$y^\nabla = \ominus_\nu(-p^\rho)y, \quad y(t_0) = 1.$$

Let $y(t) = e_p(t, t_0)$. Then

$$y(t_0) = e_p(t_0, t_0) = 1.$$

Furthermore,

$$
\begin{aligned}
y^\nabla(t) &= e_p^\nabla(t, t_0) \\
&= e_p^{\Delta\rho}(t, t_0) \\
&= p^\rho(t)e_p^\rho(t, t_0) \\
&= p^\rho(t)\left[e_p(t, t_0) - \nu(t)e_p^\nabla(t, t_0)\right].
\end{aligned}
$$

Rearranging this equation, we get

$$e_p^\nabla(t, t_0)\left[1 + \nu(t)p^\rho(t)\right] = p^\rho(t)e_p(t, t_0),$$

so

$$e_p^\nabla(t, t_0) = \frac{p^\rho(t)}{1 + \nu(t)p^\rho(t)} e_p(t, t_0).$$

Putting this back in terms of y, we get

$$y^\nabla(t) = \ominus_\nu(-p^\rho)y(t),$$

and the proof is complete.   □

Exercise 1.18. Prove the second statement of Theorem 1.17.

Example 1.19. Let $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, and let $\alpha \in \mathcal{R}_\nu$ be constant, i.e.,

$$\alpha \in \mathbb{C} \setminus \left\{ \frac{1}{h} \right\},$$

where $\mathbb{C}$ denotes the set of complex numbers. Given that

$$e_{-\alpha}(t, 0) = (1 - \alpha h)^{\frac{t}{h}},$$

use Theorem 1.17 to find $\hat{e}_\alpha(t, 0)$.

We have that $\alpha \in \mathcal{R}_\nu$, and, as α is constant, it is continuous. Therefore Theorem 1.17 applies, and we see that $-\alpha \in \mathcal{R}$ and

$$
\begin{aligned}
\hat{e}_\alpha(t, 0) &= e_{\ominus(-\alpha^\sigma)}(t, t_0) \\
&= e_{\ominus(-\alpha)}(t, t_0) \\
&= \frac{1}{e_{(-\alpha)}(t, t_0)} \\
&= \frac{1}{(1 - \alpha h)^{\frac{t}{h}}} \\
&= \left(\frac{1}{1 - \alpha h} \right)^{\frac{t}{h}}.
\end{aligned}
$$

Exercise 1.20. Let

$$\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}.$$

Given that $e_1(t, 0) = 2^{\sqrt{t}}(\sqrt{t})!$, find $\hat{e}_{-1}(t, 0)$.

Example 1.21. Suppose p is a regressive constant and q is a ν-regressive constant. Simplify the following expressions:

(i) $e_p(t, t_0)\hat{e}_q(t, t_0)$

As p and q satisfy the appropriate regressivity conditions, Theorem 1.17 applies and gives

$$
\begin{aligned}
e_p(t, t_0)\hat{e}_q(t, t_0) &= e_p(t, t_0)e_{\ominus(-q^\sigma)}(t, t_0) \\
&= e_p(t, t_0)e_{\ominus(-q)}(t, t_0) \\
&= e_{p\ominus(-q)}(t, t_0).
\end{aligned}
$$

(ii) $e_p^\nabla(t, t_0)\hat{e}_q(t, t_0)$

Again, we have that p and q satisfy the conditions of Theorem 1.17. We get

$$
\begin{aligned}
e_p^\nabla(t, t_0)\hat{e}_q(t, t_0) &= \hat{e}_{\ominus_\nu(-p^\rho)}^\nabla(t, t_0)\hat{e}_q(t, t_0) \\
&= \ominus_\nu(-p^\rho)\hat{e}_{\ominus_\nu(-p^\rho)}(t, t_0)\hat{e}_q(t, t_0) \\
&= \ominus_\nu(-p)\hat{e}_{\ominus_\nu(-p)}(t, t_0)\hat{e}_q(t, t_0) \\
&= \ominus_\nu(-p)\hat{e}_{\ominus_\nu(-p)\oplus_\nu q}(t, t_0) \\
&= \ominus_\nu(-p)\hat{e}_{q\ominus_\nu(-p)}(t, t_0).
\end{aligned}
$$

Exercise 1.22. Suppose p is a regressive constant and q is a ν-regressive constant. Simplify the following expressions:

(i) $e_p(t, t_0)\hat{e}_q^\Delta(t, t_0)$;

(ii) $\hat{e}_{\frac{p}{1+p\nu}}^\Delta(t, t_0)\hat{e}_q^\nabla(t, t_0)$;

(iii) $\left(e_p^\Delta(t, t_0)\right)^\nabla$.

3   Second-order Linear Dynamic Equations

Now, recall that we are interested in the second-order self-adjoint dynamic equation

$$
Lx = 0 \quad \text{where} \quad Lx(t) = (p(t)x^\Delta(t))^\nabla + q(t)x(t). \tag{1.1}
$$

Here we assume that $p : \mathbb{T} \to \mathbb{R}$ is continuous, $q : \mathbb{T} \to \mathbb{R}$ is ld-continuous and that

$$
p(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}.
$$

Define the set $\mathbb{D}$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is continuous and such that $[p(t)x^\Delta]^\nabla : \mathbb{T}_\kappa^\kappa \to \mathbb{R}$ is ld-continuous. A function $x \in \mathbb{D}$ is said to be a solution of $Lx = 0$ on $\mathbb{T}$ provided $Lx(t) = 0$ for all $t \in \mathbb{T}_\kappa^\kappa$.

Now, consider the second order linear dynamic equations

$$
M_1x = 0 \text{ where } M_1x = x^{\Delta\nabla} + p_1(t)x^\nabla + p_2(t)x, \tag{1.2}
$$

$$
M_2x = 0 \text{ where } M_2x = x^{\Delta\nabla} + a_1(t)x^\Delta + a_2(t)x, \tag{1.3}
$$

and

$$
M_3x = 0 \text{ where } M_3x = x^{\Delta\nabla} + r_1(t)x^\nabla + r_2(t)x^\rho, \tag{1.4}
$$

where $p_i, a_i, r_i : \mathbb{T} \to \mathbb{R}$ are ld-continuous for $i \in \{1, 2\}$. Take $\mathbb{D}_M$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that x is Δ-differentiable on $\mathbb{T}^\kappa$, $x^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_\kappa^\kappa$, and $x^{\Delta\nabla} : \mathbb{T}_\kappa^\kappa \to \mathbb{R}$ is ld-continuous. For $i = 1, 2, 3$, we say x is a solution of $M_ix = 0$ on $\mathbb{T}$ provided x is in $\mathbb{D}_M$, and $M_ix = 0$ for all $t \in \mathbb{T}_\kappa^\kappa$.

Theorem 1.23. *If p_2 is ld-continuous and $p_1 \in \mathcal{R}_\nu^+$, then the dynamic equation* (1.2) *can be written in self-adjoint form, with*

$$p(t) = \hat{e}_{p_1}(t, t_0) \text{ and } q(t) = \hat{e}_{p_1}(t, t_0)p_2(t).$$

Furthermore, in this case, if x is a solution of (1.2) *on $\mathbb{T}$, then x is also a solution of the self-adjoint form of the equation.*

Proof. Suppose we have

$$x^{\Delta\nabla} + p_1(t)x^\nabla + p_2(t)x = 0.$$

Assume p_2 is ld-continuous and $p_1 \in \mathcal{R}_\nu^+$. Then $\hat{e}_{p_1}(t, t_0)$ is well defined and positive. Multiplying through by $\hat{e}_{p_1}(t, t_0)$, we get

$$\hat{e}_{p_1}(t, t_0)x^{\Delta\nabla} + \hat{e}_{p_1}(t, t_0)p_1(t)x^\nabla + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

Then, since $\hat{e}_{p_1}(t, t_0)$ solves the IVP

$$y^\nabla = p_1(t)y, \quad y(t_0) = 1,$$

we have that

$$[\hat{e}_{p_1}(t, t_0)]^\nabla = p_1(t)\hat{e}_{p_1}(t, t_0).$$

So our equation becomes

$$\hat{e}_{p_1}(t, t_0)x^{\Delta\nabla} + [\hat{e}_{p_1}(t, t_0)]^\nabla x^\nabla + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

Furthermore, x^Δ is ∇-differentiable, hence continuous, so by Corollary 1.11, $x^\nabla = x^{\Delta\rho}$ and we get

$$\hat{e}_{p_1}(t, t_0)x^{\Delta\nabla} + [\hat{e}_{p_1}(t, t_0)]^\nabla x^{\Delta\rho} + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

Then by the product rule, we see that

$$[\hat{e}_{p_1}(t, t_0)x^\Delta]^\nabla + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

This equation is in self-adjoint form with $p(t)$ and $q(t)$ as desired.

Now suppose x is a solution of (1.2), p_2 is ld-continuous and $p_1 \in \mathcal{R}_\nu^+$. Based on the above development, it is clear that x satisfies the dynamic equation

$$[\hat{e}_{p_1}(t, t_0)x^\Delta]^\nabla + \hat{e}_{p_1}(t, t_0)p_2(t)x = 0.$$

Hence to show x is a solution of this dynamic equation, we need only show that $x \in \mathbb{D}$. Note that x^Δ is ∇-differentiable, and therefore continuous. Also,

$$[\hat{e}_{p_1}(t, t_0)x^\Delta]^\nabla = \hat{e}_{p_1}^\nabla(t, t_0)x^\rho + \hat{e}_{p_1}(t, t_0)x^{\Delta\nabla} = p_1(t)\hat{e}_{p_1}(t, t_0)x^\rho + \hat{e}_{p_1}(t, t_0)x^{\Delta\nabla},$$

which is ld-continuous, and therefore, $x \in \mathbb{D}$. $\square$

Corollary 1.24. *If a_2 is ld-continuous and $-a_1 \in \mathcal{R}_\nu^+$, then the dynamic equation (1.3) can be written in self-adjoint form, with*

$$p(t) = \hat{e}_{\frac{a_1}{1+a_1\nu}}(t, t_0) \quad and \quad q(t) = \frac{a_2(t)}{1 + a_1(t)\nu(t)} \hat{e}_{\frac{a_1}{1+a_1\nu}}(t, t_0).$$

Furthermore, if x is a solution of (1.3), then x is also a solution of the self-adjoint form of the equation.

Proof. Suppose we have

$$x^{\Delta\nabla} + a_1(t)x^\Delta + a_2(t)x = 0.$$

Recall that if $f : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable, then $f(t) = f^\rho(t) + \nu(t)f^\nabla(t)$. Thus $x^\Delta = x^{\Delta\rho} + \nu(t)x^{\Delta\nabla}$. Making this substitution, we have

$$x^{\Delta\nabla} + a_1(t)(x^{\Delta\rho} + \nu(t)x^{\Delta\nabla}) + a_2(t)x = 0,$$

and hence

$$(1 + a_1(t)\nu(t))x^{\Delta\nabla} + a_1(t)x^{\Delta\rho} + a_2(t)x = 0.$$

Now $-a_1(t) \in \mathcal{R}_\nu^+$, so the leading coefficient is positive and we may divide through by it. Furthermore, as before, we have that x^Δ is continuous, so $x^\nabla = x^{\Delta\rho}$. Thus, we get

$$x^{\Delta\nabla} + \frac{a_1(t)}{(1 + a_1(t)\nu(t))}x^\nabla + \frac{a_2(t)}{(1 + a_1(t)\nu(t))}x = 0.$$

This is in the form (1.2). As a_1 and a_2 are ld-continuous, so are $\frac{a_1}{(1+a_1\nu)}$ and $\frac{a_2}{(1+a_1\nu)}$. Further,

$$\left(1 - \nu(t)\frac{a_1(t)}{1 + a_1(t)\nu(t)}\right) = \frac{1 + a_1(t)\nu(t) - a_1(t)\nu(t)}{1 + a_1(t)\nu(t)} = \frac{1}{1 + a_1(t)\nu(t)} > 0,$$

so the coefficient of the x^∇ term is in $\mathcal{R}_\nu^+$. Hence by Theorem 1.23 above, the equation can be written in self-adjoint form, with $p(t)$ and $q(t)$ in the desired form, and solutions of equation (1.3) are also solutions of the self-adjoint form of the equation. □

Corollary 1.25. *If r_2 is ld-continuous and $(r_1 - \nu r_2) \in \mathcal{R}_\nu^+$, then the dynamic equation (1.4) can be written in self-adjoint form, with*

$$p(t) = \hat{e}_{(r_1 - \nu r_2)}(t, t_0) \quad and \quad q(t) = r_2(t)\hat{e}_{(r_1 - \nu r_2)}(t, t_0).$$

Furthermore, if x is a solution of (1.4), then x is also a solution of the self-adjoint form of the equation.

Proof. Suppose we have

$$x^{\Delta\nabla} + r_1(t)x^{\nabla} + r_2(t)x^{\rho} = 0.$$

Then

$$x^{\Delta\nabla} + r_1(t)x^{\nabla} + r_2(t)(x - \nu(t)x^{\nabla}) = 0,$$

or

$$x^{\Delta\nabla} + (r_1(t) - \nu(t)r_2(t))x^{\nabla} + r_2(t)x = 0.$$

This is in the form (1.2), and the coefficients meet the requirements of Theorem 1.23. Thus the result follows. □

Example 1.26. Write the following dynamic equations in self-adjoint form:

(i) $3x^{\Delta\nabla} - 5x^{\nabla} + 6x = 0$

This dynamic equation is in the form of M_1x. To find p_1 and p_2, we need to divide through by the leading coefficient. This gives us $p_1(t) = -\frac{5}{3}$ and $p_2(t) = 2$. Both p_1 and p_2 are continuous and $p_1 \in \mathcal{R}_\nu^+$. We then apply Theorem 1.23 to get the self-adjoint dynamic equation

$$\left[\hat{e}_{(-\frac{5}{3})}(t, t_0)x^{\Delta}\right]^{\nabla} + 2\hat{e}_{(-\frac{5}{3})}(t, t_0)x = 0.$$

(ii) $4x^{\Delta\nabla} + x^{\Delta} + 3x = 0$

This dynamic equation is in the form of M_2x. To find a_1 and a_2, we need to divide through by the leading coefficient. This gives us $a_1(t) = \frac{1}{4}$, and $a_2(t) = \frac{3}{4}$. Both a_1 and a_2 are continuous, and $-a_1 \in \mathcal{R}_\nu^+$. We then apply Corollary 1.24 to get the self-adjoint dynamic equation

$$\left[\hat{e}_{\frac{1}{4+\nu}}(t, t_0)x^{\Delta}\right]^{\nabla} + \left(\frac{3}{4 + \nu(t)}\right)\hat{e}_{\frac{1}{4+\nu}}(t, t_0)x = 0.$$

Exercise 1.27. Write the following dynamic equations in self-adjoint form:

(i) $x^{\Delta\nabla} - 2x^{\nabla} + x = 0$;

(ii) $x^{\Delta\nabla} + 4x = 0$;

(iii) $3x^{\Delta\nabla} + 4x^{\Delta} - 2x = 0$;

(iv) $5x^{\Delta\nabla} - x^{\nabla} + 2x^{\rho} = 0$.

We now seek to develop techniques for solving second-order linear dynamic equations of the form

$$x^{\Delta\nabla} + \alpha x^{\nabla} + \beta x^{\rho} = 0,$$

where α and β are real constants. As one would expect, we will use a characteristic equation, and look for exponential solutions. In order for the roots of our characteristic equation to be regressive, we impose the regressivity condition $1 - \alpha\nu(t) + \beta\nu^2(t) \neq 0$.

Theorem 1.28. *Let $\alpha, \beta \in \mathbb{R}$ with $1 - \alpha\nu(t) + \beta\nu^2(t) \neq 0$, and let*

$$Mx := x^{\Delta\nabla} + \alpha x^{\nabla} + \beta x^{\rho}.$$

Furthermore, suppose the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0$$

has distinct real roots, λ_1 and λ_2. Then a general solution of $Mx = 0$ is given by

$$x(t) = c_1 e_{\lambda_1}(t, t_0) + c_2 e_{\lambda_2}(t, t_0).$$

Proof. We first show that λ_1 and λ_2 are regressive. By Lemma 1.15, it suffices to show that $-\lambda_1$ and $-\lambda_2$ are ν-regressive. So, consider

$$
\begin{aligned}
(1 - (-\lambda_1)\nu(t))(1 - (-\lambda_2)\nu(t) &= (1 + \lambda_1\nu(t))(1 + \lambda_2\nu(t)) \\
&= 1 + (\lambda_1 + \lambda_2)\nu(t) + (\lambda_1\lambda_2)\nu^2(t) \\
&= 1 - \alpha\nu(t) + \beta\nu^2(t) \\
&\neq 0.
\end{aligned}
$$

Thus λ_1 and λ_2 are regressive, thus the exponential functions $e_{\lambda_1}(t, t_0)$ and $e_{\lambda_2}(t, t_0)$ are well defined and in the domain of our operator M.

Now, as $\lambda_1 \neq \lambda_2$, we have that $e_{\lambda_1}(t, t_0)$ and $e_{\lambda_2}(t, t_0)$ are linearly independent. Thus by Theorem 1.39, which will be stated in the next section, it suffices to show that $e_{\lambda_1}(t, t_0)$ and $e_{\lambda_2}(t, t_0)$ are both solutions of $Mx = 0$. We will only show that $e_{\lambda_1}(t, t_0)$ is a solution, as the other part of the proof is similar. Let $y(t) := e_{\lambda_1}(t, t_0)$. Then Theorem 1.17 and Corollary 1.11 apply, and we see that

$$y^{\Delta}(t) = \lambda_1 e_{\lambda_1}(t, t_0), \quad \text{and} \quad y^{\nabla}(t) = \frac{\lambda_1}{1 + \lambda_1\nu(t)} e_{\lambda_1}(t, t_0).$$

We now substitute into our dynamic equation and get

$$
\begin{aligned}
My &= (\lambda_1 e_{\lambda_1}(t, t_0))^{\nabla} + \alpha \frac{\lambda_1}{1 + \lambda_1\nu(t)} e_{\lambda_1}(t, t_0) + \beta e_{\lambda_1}^{\rho}(t, t_0) \\
&= \frac{\lambda_1^2}{1 + \lambda_1\nu(t)} e_{\lambda_1}(t, t_0) + \frac{\alpha\lambda_1}{1 + \lambda_1\nu(t)} e_{\lambda_1}(t, t_0) \\
&\quad + \beta \left[e_{\lambda_1}(t, t_0) - \nu(t) \frac{\lambda_1}{1 + \lambda_1\nu(t)} e_{\lambda_1}(t, t_0) \right] \\
&= \frac{e_{\lambda_1}(t, t_0)}{1 + \lambda_1\nu(t)} \left[\lambda_1^2 + \alpha\lambda_1 + \beta(1 + \lambda_1\nu(t)) - \beta\lambda_1\nu(t) \right] \\
&= \frac{e_{\lambda_1}(t, t_0)}{1 + \lambda_1\nu(t)} \left[\lambda_1^2 + \alpha\lambda_1 + \beta \right] \\
&= 0.
\end{aligned}
$$

Similarly $e_{\lambda_2}(t, t_0)$ is a solution of $Mx = 0$. Hence $x(t) = c_1 e_{\lambda_1}(t, t_0) + c_2 e_{\lambda_2}(t, t_0)$ is our general solution, as desired. $\qquad\square$

Example 1.29. Solve the following dynamic equations:

(i) $x^{\Delta\nabla} - 8x^{\nabla} + 15x^{\rho} = 0$

This dynamic equation is in the form $Mx = 0$, where

$$Mx := x^{\Delta\nabla} + \alpha x^{\nabla} + \beta x^{\rho}.$$

In this case, we have $\alpha = -8$ and $\beta = 15$ and

$$1 - \alpha\nu(t) + \beta\nu^2(t) = 1 + 8\nu(t) + 15\nu^2(t) > 0.$$

Thus by Theorem 1.28 we need to find the roots of the "characteristic equation"

$$\lambda^2 - 8\lambda + 15 = 0.$$

This factors nicely, and we get

$$(\lambda - 3)(\lambda - 5) = 0,$$

so our roots are $\lambda_1 = 3$ and $\lambda_2 = 5$. Applying the theorem, we see that our general solution is

$$x(t) = c_1 e_3(t, t_0) + c_2 e_5(t, t_0).$$

(ii) $x^{\Delta\nabla} - 5x^{\nabla} + 6x^{\rho} = 0$

This time we have $\alpha = -5$ and $\beta = 6$, and

$$1 - \alpha\nu(t) + \beta\nu^2(t) = 1 + 5\nu(t) + 6\nu^2(t) > 0.$$

So our "characteristic equation" is

$$\lambda^2 - 5\lambda + 6 = 0,$$

which factors into

$$(\lambda - 2)(\lambda - 3) = 0,$$

so our roots are $\lambda_1 = 2$ and $\lambda_2 = 3$. Applying the theorem, we see that our general solution is

$$x(t) = c_1 e_2(t, t_0) + c_2 e_3(t, t_0).$$

Exercise 1.30. Solve the following dynamic equations:

(i) $x^{\Delta\nabla} - 12x^{\nabla} + 11x^{\rho} = 0$;

(ii) $x^{\Delta\nabla} - 6x^{\nabla} + 8x^{\rho} = 0$;

(iii) $x^{\Delta\nabla} - 7x^{\nabla} + 12x^{\rho} = 0$.

4   Abel's Formula and Reduction of Order

We begin this section by looking at the Lagrange Identity for the dynamic equation (1.1). We establish several corollaries and related results, including Abel's Formula and its converse. We conclude the section with a reduction of order theorem. Some of the results in this section are due to Atici and Guseinov. Specifically, Theorems 1.31 and 1.39, and Corollaries 1.35 and 1.38 were previously established in their work [20].

Theorem 1.31. *If $t_0 \in \mathbb{T}$, and x_0 and x_1 are given constants, then the initial value problem*

$$Lx = 0, \quad x(t_0) = x_0, \quad x^\Delta(t_0) = x_1$$

has a unique solution, and this solution exists on all of $\mathbb{T}$.

Definition 1.32. *If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the Wronskian of x and y, denoted $W(x,y)(t)$ is defined by*

$$W(x,y)(t) = \begin{vmatrix} x(t) & y(t) \\ x^\Delta(t) & y^\Delta(t) \end{vmatrix} \quad \text{for } t \in \mathbb{T}^\kappa.$$

Definition 1.33. *If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the Lagrange bracket of x and y is defined by*

$$\{x; y\}(t) = p(t)W(x,y)(t) \quad \text{for } t \in \mathbb{T}^\kappa.$$

Theorem 1.34 (Lagrange Identity). *If $x, y \in \mathbb{D}$, then*

$$x(t)Ly(t) - y(t)Lx(t) = \{x; y\}^\nabla(t) \quad \text{for } t \in \mathbb{T}^\kappa_\kappa.$$

Proof. Let $x, y \in \mathbb{D}$. We have

$$
\begin{aligned}
\{x; y\}^\nabla &= [pW(x,y)]^\nabla \\
&= [xpy^\Delta - ypx^\Delta]^\nabla \\
&= x^\nabla p^\rho y^{\Delta\rho} + x[py^\Delta]^\nabla - y^\nabla p^\rho x^{\Delta\rho} - y[px^\Delta]^\nabla \\
&= x^\nabla p^\rho y^\nabla + x[py^\Delta]^\nabla - y^\nabla p^\rho x^\nabla - y[px^\Delta]^\nabla \\
&= x[py^\Delta]^\nabla - y[px^\Delta]^\nabla \\
&= x([py^\Delta]^\nabla + qy) - y([px^\Delta]^\nabla + qx) \\
&= xLy - yLx,
\end{aligned}
$$

where we have made use of the fact that x^Δ and y^Δ are continuous and applied Corollary 1.11.   □

Corollary 1.35 (Abel's Formula). *If x, y are solutions of (1.1), then*

$$W(x,y)(t) = \frac{C}{p(t)} \quad \text{for } t \in \mathbb{T}^\kappa,$$

where C is a constant.

Proof. If x, y are solutions of (1.1), they belong to $\mathbb{D}$. Then, by Theorem 1.34, we have

$$x(t)Ly(t) - y(t)Lx(t) = \{x; y\}^{\nabla}(t) \quad \text{for } t \in \mathbb{T}^{\kappa}.$$

But $Lx = Ly = 0$, so

$$0 = \{x; y\}^{\nabla}(t) \quad \text{for } t \in \mathbb{T}^{\kappa}.$$

Integrating, we see that

$$\{x; y\} = p(t)W(x, y)(t) = C,$$

which gives the desired result. $\qquad\square$

Definition 1.36. Define the *inner product* of x and y on $[a, b]$ by

$$\langle x, y \rangle := \int_a^b x(t)y(t)\nabla t.$$

Corollary 1.37 (Green's Formula). *If* $x, y \in \mathbb{D}$, *then*

$$\langle x, Ly \rangle - \langle Lx, y \rangle = [p(t)W(x, y)]_a^b.$$

Proof. Integrating the expression in Theorem 1.34 gives the result immediately. $\qquad\square$

Corollary 1.38. *If* x, y *are solutions of* (1.1), *then either*

(i) $W(x, y) \neq 0$ *for* $t \in \mathbb{T}^{\kappa}$ *or*

(ii) $W(x, y) \equiv 0$ *for* $t \in \mathbb{T}^{\kappa}$.

Case (i) occurs if and only if x and y are linearly independent on $\mathbb{T}$, and case (ii) occurs if and only if x and y are linearly dependent on $\mathbb{T}$.

In the standard way, one uses the uniqueness theorem to prove the following result.

Theorem 1.39. *If x_1 and x_2 are linearly independent solutions of* (1.1) *on* $\mathbb{T}$, *then a general solution of* (1.1) *is given by*

$$x(t) = c_1 x_1(t) + c_2 x_2(t).$$

So, we see that, as we would expect with a second-order dynamic equation, we need only find two linearly independent solutions in order to construct a general solution. We now turn our attention to some results that will assist us in actually *finding* these solutions.

Theorem 1.40 (Converse of Abel's Formula). *Assume u is a solution of* (1.1) *with $u(t) \neq 0$ for $t \in \mathbb{T}$. If $v \in \mathbb{D}$ satisfies*

$$W(u, v)(t) = \frac{C}{p(t)},$$

then v is also a solution of (1.1).

Proof. Suppose that u is a solution of (1.1) with $u(t) \neq 0$ for any t, and assume that $v \in \mathbb{D}$ satisfies $W(u,v)(t) = \frac{C}{p(t)}$. Then by Theorem 1.34, we have

$$u(t)Lv(t) - v(t)Lu(t) = \{u;v\}^{\nabla}(t),$$

so

$$
\begin{aligned}
u(t)Lv(t) &= [p(t)W(u,v)(t)]^{\nabla} \\
&= [p(t)\frac{C}{p(t)}]^{\nabla} \\
&= C^{\nabla} \\
&= 0.
\end{aligned}
$$

As $u(t) \neq 0$ for any t, we can divide through by it to get

$$Lv(t) = 0 \quad \text{for } t \in \mathbb{T}_{\kappa}^{\kappa}.$$

Hence v is a solution of (1.1) on $\mathbb{T}$. $\square$

Theorem 1.41 (Reduction of Order). *Let $t_0 \in \mathbb{T}^{\kappa}$, and assume u is a solution of (1.1) with $u(t) \neq 0$ for any t. Then a second, linearly independent solution, v, of (1.1) is given by*

$$v(t) = u(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s$$

for $t \in \mathbb{T}$.

Proof. By Theorem 1.40, we need only show that $v \in \mathbb{D}$ and that $W(u,v)(t) = \frac{C}{p(t)}$ for some constant C. Consider first

$$
\begin{aligned}
W(u,v)(t) &= \begin{vmatrix} u(t) & v(t) \\ u^{\Delta}(t) & v^{\Delta}(t) \end{vmatrix} \\
&= u(t)v^{\Delta}(t) - v(t)u^{\Delta}(t) \\
&= u(t)\left[u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s + \frac{u^{\sigma}(t)}{p(t)u(t)u^{\sigma}(t)} \right] \\
&\quad - u^{\Delta}(t)u(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s \\
&= u(t)u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s + \frac{u(t)u^{\sigma}(t)}{p(t)u(t)u^{\sigma}(t)} \\
&\quad - u(t)u^{\Delta}(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s \\
&= \frac{1}{p(t)}.
\end{aligned}
$$

Here we have $C = 1$. It remains to show that $v \in \mathbb{D}$. We have that

$$
\begin{aligned}
v^\Delta(t) &= u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s + \frac{u^\sigma(t)}{p(t)u(t)u^\sigma(t)} \\
&= u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s + \frac{1}{p(t)u(t)}.
\end{aligned}
$$

Since $u \in \mathbb{D}, u(t) \neq 0$ and p is continuous, we have that v^Δ is continuous. Next, consider

$$
\begin{aligned}
[p(t)v^\Delta(t)]^\nabla &= \left[p(t)u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s \right]^\nabla + \left[\frac{1}{u(t)} \right]^\nabla \\
&= [p(t)u^\Delta(t)]^\nabla \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s \\
&\quad + p^\rho(t)u^{\Delta\rho}(t) \left[\int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s \right]^\nabla - \frac{u^\Delta(t)}{u(t)u^\rho(t)}.
\end{aligned}
$$

Now, the first and last terms are ld-continuous. It is not as clear that the center term is ld-continuous. Specifically, we are concerned about whether or not the expression

$$
\left[\int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s \right]^\nabla
$$

is ld-continuous. Note that the integrand is rd-continuous. Hence Theorem 1.10 applies and yields

$$
\left[\int_{t_0}^t \frac{1}{p(\tau)u(\tau)u^\sigma(\tau)} \Delta \tau \right]^\nabla = \begin{cases} \frac{1}{p^\rho(t)u^\rho(t)u^{\sigma\rho}(t)} & t \in \mathbb{T}_A \\[2mm] \lim_{s \to t-} \frac{1}{p(s)u(s)u^\sigma(s)} & t \in A. \end{cases}
$$

First, suppose $t \in \mathbb{T}_A$. Then $\sigma(\rho(t)) = t$, so our expression simplifies to

$$
\left[\int_{t_0}^t \frac{1}{p(\tau)u(\tau)u^\sigma(\tau)} \Delta \tau \right]^\nabla = \frac{1}{p^\rho(t)u^\rho(t)u^{\sigma\rho}(t)} = \frac{1}{p^\rho(t)u^\rho(t)u(t)}.
$$

Next, suppose that $t \in A$. Then t is left-dense, and therefore,

$$
\lim_{s \to t-} \sigma(s) = t.
$$

Then as p and u are continuous, we have

$$
\lim_{s \to t-} \frac{1}{p(s)u(s)u^\sigma(s)} = \frac{1}{p(t)u^2(t)}.
$$

Now for $t \in A$, t is left-dense, so, if we like, we may write this expression as

$$\lim_{s \to t-} \frac{1}{p(s)u(s)u^{\sigma}(s)} = \frac{1}{p(t)u^2(t)} = \frac{1}{p^{\rho}(t)u^{\rho}(t)u(t)}.$$

This is the same expression we got for $t \in \mathbb{T}_A$, so we have that

$$\left[\int_{t_0}^{t} \frac{1}{p(\tau)u(\tau)u^{\sigma}(\tau)} \Delta\tau \right]^{\nabla} = \frac{1}{p^{\rho}(t)u^{\rho}(t)u(t)} \quad \text{for } t \in \mathbb{T}.$$

This function is ld-continuous, and so we have that $v \in \mathbb{D}$. Hence by Theorem 1.40, v is also a solution of (1.1). Finally, note that as $W(u,v)(t) = \frac{1}{p(t)} \neq 0$ for any t, u and v are linearly independent. $\qquad\square$

The following example and exercise illustrate the use of the reduction of order theorem. The reader should be aware, however, that they rely heavily on properties of the generalized exponential functions which are not discussed here. If additional background on the generalized exponential functions is desired, we refer the reader to Chapters 1 and 3.

Example 1.42. Given that $e_1(t, t_0)$ solves the dynamic equation

$$x^{\Delta\nabla} - 3x^{\nabla} + 2x^{\rho} = 0,$$

use Reduction of Order to find a second linearly independent solution.

To use Theorem 1.41, we must first put our dynamic equation in self-adjoint form. We get

$$\left[\hat{e}_{(-3-2\nu)}(t, t_0)x^{\Delta} \right]^{\nabla} + 2\hat{e}_{(-3-2\nu)}(t, t_0)x = 0.$$

So $p(t) = \hat{e}_{(-3-2\nu)}(t, t_0)$. Let $u(t) = e_1(t, t_0)$. Then by Theorem 1.41, our second, linearly independent solution, v is given by

$$\begin{aligned}
v(t) &= u(t) \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \Delta s \\
&= e_1(t, t_0) \int_{t_0}^{t} \frac{1}{\hat{e}_{(-3-2\nu)}(s, t_0)e_1(s, t_0)e_1^{\sigma}(s, t_0)} \Delta s.
\end{aligned}$$

We now wish to simplify the denominator of the integrand. Note that

$$-1 \oplus_v -2 = -1 + (-2) - \nu(t)(-1)(-2) = -3 - 2\nu(t),$$

and thus

$$\hat{e}_{(-3-2\nu)}(t, t_0) = \hat{e}_{-1}(t, t_0)\hat{e}_{-2}(t, t_0).$$

Applying Theorem 1.17 to each term of this product separately, we see that

$$\hat{e}_{-1}(t, t_0) = e_{\ominus 1}(t, t_0) \quad \text{and} \quad \hat{e}_{-2}(t, t_0) = e_{\ominus 2}(t, t_0).$$

We will also move these terms to the numerator, to get

$$
\begin{aligned}
v(t) &= e_1(t, t_0) \int_{t_0}^{t} \frac{e_1(s, t_0) e_2(s, t_0)}{e_1(s, t_0) e_1^{\sigma}(s, t_0)} \, \Delta s \\
&= e_1(t, t_0) \int_{t_0}^{t} \frac{e_2(s, t_0)}{e_1^{\sigma}(s, t_0)} \, \Delta s \\
&= e_1(t, t_0) \int_{t_0}^{t} e_{2 \ominus 1}^{\Delta}(s, t_0) \, \Delta s \\
&= e_1(t, t_0) \left[e_{2 \ominus 1}(t, t_0) - e_{2 \ominus 1}(t_0, t_0) \right] \\
&= e_1(t, t_0) e_{2 \ominus 1}(t, t_0) - e_1(t, t_0) \\
&= e_{1 \oplus 2 \ominus 1}(t, t_0) - e_1(t, t_0) \\
&= e_2(t, t_0) - e_1(t, t_0).
\end{aligned}
$$

Exercise 1.43. For each of the following, given that $u(t)$ solves the given dynamic equation, use reduction of order to find a second, linearly independent solution.

(i) $u(t) = e_1(t, t_0);$ $x^{\Delta \nabla} - 4x^{\nabla} + 3x^{\rho} = 0;$

(ii) $u(t) = e_2(t, t_0);$ $x^{\Delta \nabla} - 8x^{\nabla} + 12x^{\rho} = 0;$

(iii) $u(t) = e_5(t, t_0);$ $x^{\Delta \nabla} - 7x^{\nabla} + 10x^{\rho} = 0;$

5 Oscillation and Disconjugacy

In this section, we establish results concerning generalized zeros of solutions of (1.1), and examine disconjugacy and oscillation of solutions.

Definition 1.44. We say that a solution, x, of (1.1) has a *generalized zero* at t if

$$
x(t) = 0
$$

or, if t is left-scattered and

$$
x(\rho(t)) x(t) < 0.
$$

Definition 1.45. We say that (1.1) is *disconjugate* on an interval $[a, b]$ if the following hold.

(i) If x is a nontrivial solution of (1.1) with $x(a) = 0$, then x has no generalized zeros in $(a, b]$.

(ii) If x is a nontrivial solution of (1.1) with $x(a) \neq 0$, then x has at most one generalized zero in $(a, b]$.

Definition 1.46. Let $\omega = \sup \mathbb{T}$, and if $\omega < \infty$, assume $\rho(\omega) = \omega$. Let $a \in \mathbb{T}$. We say that (1.1) is *oscillatory* on $[a, \omega)$ if every nontrivial real-valued solution has infinitely many generalized zeros in $[a, \omega)$. We say (1.1) is *nonoscillatory* on $[a, \omega)$ if it is not oscillatory on $[a, \omega)$.

Lemma 1.47. *Let $\omega = \sup \mathbb{T}$. If $\omega < \infty$, then assume $\rho(\omega) = \omega$. Let $a \in \mathbb{T}$. Then if (1.1) is nonoscillatory on $[a, \omega)$, there is some $t_0 \in \mathbb{T}$, $t_0 \geq a$, such that (1.1) has a positive solution on $[t_0, \omega)$.*

Proof. Assume (1.1) is nonoscillatory on $[a, \omega)$, and then there is a nontrivial solution, u of (1.1) such that u has only finitely many generalized zeros in $[a, \omega)$. Let $b = \max\{t \in \mathbb{T} : u \text{ has a generalized zero at } t\}$. Fix $t_0 \in \mathbb{T}$ such that $t_0 > b$. Then either $u > 0$ on $[t_0, \omega)$ or $-u > 0$ on $[t_0, \omega)$. $\qquad\square$

Our first oscillation theorem is the Sturm Separation Theorem. Loosely speaking, this theorem tells us that (generalized) zeros of linearly independent solutions of (1.1) separate one another. Thus we see that either all solutions of $Lx = 0$ will be oscillatory or they will all be nonoscillatory.

Theorem 1.48 (Sturm Separation Theorem). *Let u and v be linearly independent solutions of (1.1) on $\mathbb{T}$. Then u and v have no common zeros in $\mathbb{T}^\kappa$. If u has a zero at $t_1 \in \mathbb{T}$, and a generalized zero at $t_2 > t_1 \in \mathbb{T}$, then v has a generalized zero in $(t_1, t_2]$. If u has generalized zeros at $t_1 \in \mathbb{T}$ and $t_2 > t_1 \in \mathbb{T}$, then v has a generalized zero in $[t_1, t_2]$.*

Proof. If u and v have a common zero at $t_0 \in \mathbb{T}^\kappa$, then

$$W(u,v)(t_0) = \begin{vmatrix} u(t_0) & v(t_0) \\ u^\Delta(t_0) & v^\Delta(t_0) \end{vmatrix} = 0.$$

Hence u and v are linearly dependent.

Now suppose u has a zero at $t_1 \in \mathbb{T}$, and a generalized zero at $t_2 > t_1 \in \mathbb{T}$. Without loss of generality, we may assume $t_2 > \sigma(t_1)$ is the first generalized zero to the right of t_1, $u(t) > 0$ on (t_1, t_2), and $u(t_2) \leq 0$. Assume v is a linearly independent solution of (1.1) with no generalized zero in $(t_1, t_2]$. Without loss of generality, $v(t) > 0$ on $[t_1, t_2]$.

Then on $[t_1, t_2]$,

$$\left(\frac{u}{v}\right)^\Delta (t) = \frac{v(t)u^\Delta(t) - u(t)v^\Delta(t)}{v(t)v^\sigma(t)} = \frac{C}{p(t)v(t)v^\sigma(t)},$$

which is of one sign on $[t_1, t_2)$. Thus $\frac{u}{v}$ is monotone on $[t_1, t_2]$. Fix $t_3 \in (t_1, t_2)$. Note that

$$\frac{u(t_1)}{v(t_1)} = 0, \text{ and } \frac{u(t_3)}{v(t_3)} > 0.$$

But

$$\frac{u(t_2)}{v(t_2)} \leq 0,$$

which contradicts the fact that $\frac{u}{v}$ is monotone on $[t_1, t_2]$. Hence v must have a generalized zero in $(t_1, t_2]$.

Finally, suppose u has generalized zeros at $t_1 \in \mathbb{T}$ and $t_2 > t_1 \in \mathbb{T}$. Assume $t_2 > \sigma(t_1)$ is the first generalized zero to the right of t_1. If $u(t_1) = 0$, we are in the previous case, so assume $u(t_1) \neq 0$. Then, as u has a generalized zero at t_1, we have that t_1 is left-scattered. Without loss of generality, we may assume $u(t) > 0$ on $[t_1, t_2)$, $u(\rho(t_1)) < 0$ and $u(t_2) \leq 0$. Assume v is a linearly independent solution of (1.1) with no generalized zero in $[t_1, t_2]$. Without loss of generality, $v(t) > 0$ on $[t_1, t_2]$, and $v(\rho(t_1)) > 0$. In a similar fashion to the previous case, we apply Abel's Formula to get that $\frac{u}{v}$ is monotone on $[\rho(t_1), t_2]$. But

$$\frac{u(\rho(t_1))}{v(\rho(t_1))} < 0, \quad \frac{u(t_1)}{v(t_1)} > 0, \quad \text{and} \quad \frac{u(t_2)}{v(t_2)} \leq 0,$$

which is a contradiction. Hence v must have a generalized zero in $[t_1, t_2]$.   □

Theorem 1.49. *If (1.1) has a positive solution on an interval $\mathcal{I} \subset \mathbb{T}$, then (1.1) is disconjugate on $\mathcal{I}$. Conversely, if $a, b \in \mathbb{T}_\kappa^\kappa$ and (1.1) is disconjugate on $[\rho(a), \sigma(b)] \subset \mathbb{T}$, then (1.1) has a positive solution on $[\rho(a), \sigma(b)]$.*

Proof. Assume (1.1) has a positive solution, u on $\mathcal{I} \subset \mathbb{T}$. If (1.1) is not disconjugate on $\mathcal{I}$, then (1.1) has a nontrivial solution v with at least two generalized zeros in $\mathcal{I}$. Then, without loss of generality, there are t_1, t_2 in $\mathcal{I}$ such that

$$v(t_1) \leq 0, v(t_2) \leq 0, \text{ and } v(t) > 0 \text{ on } (t_1, t_2) \text{ with } (t_1, t_2) \neq \emptyset.$$

Note that

$$
\begin{aligned}
\left(\frac{v}{u}\right)^\Delta (t) &= \frac{u(t)v^\Delta(t) - v(t)u^\Delta(t)}{u(t)u^\sigma(t)} \\
&= \frac{W(u,v)(t)}{u(t)u^\sigma(t)} \\
&= \frac{C}{p(t)u(t)u^\sigma(t)}
\end{aligned}
$$

is of one sign on $\mathcal{I}^\kappa$. Hence $\frac{v}{u}$ is monotone on $\mathcal{I}$. But

$$\left(\frac{v}{u}\right)(t_1) \leq 0, \left(\frac{v}{u}\right)(t) > 0, \text{ and } \left(\frac{v}{u}\right)(t_2) \leq 0.$$

This contradicts the fact that $\frac{v}{u}$ is monotone. Hence (1.1) is disconjugate on $\mathcal{I}$.

Conversely, suppose that (1.1) is disconjugate on the compact interval $[\rho(a), \sigma(b)]$. Let u, v be the solutions of (1.1) satisfying $u(\rho(a)) = 0, u^\Delta(\rho(a)) = 1$

and $v(\sigma(b)) = 0, v^{\Delta}(b) = -1$. Since (1.1) is disconjugate on $[\rho(a), \sigma(b)]$, we have that $u(t) > 0$ on $(\rho(a), \sigma(b)]$, and $v(t) > 0$ on $[\rho(a), \sigma(b))$. Then

$$x(t) = u(t) + v(t)$$

is the desired positive solution.   □

Theorem 1.50 (Polya Factorization). *If* (1.1) *has a positive solution, u, on an interval $\mathcal{I} \subset \mathbb{T}$, then for any $x \in \mathbb{D}$, we get the* Polya *Factorization*

$$Lx = \alpha_1(t)\{\alpha_2[\alpha_1 x]^{\Delta}\}^{\nabla}(t) \ for \ t \in \mathcal{I},$$

where

$$\alpha_1 := \frac{1}{u} > 0 \quad on \ \mathcal{I},$$

and

$$\alpha_2 := puu^{\sigma} > 0 \quad on \ \mathcal{I}.$$

Proof. Assume that u is a positive solution of (1.1) on $\mathcal{I}$, and let $x \in \mathbb{D}$. Then by the Lagrange Identity (Theorem 1.34),

$$
\begin{aligned}
u(t)Lx(t) - x(t)Lu(t) &= \{u; x\}^{\nabla}(t) \\
u(t)Lx(t) &= \{u; x\}^{\nabla}(t) \\
Lx(t) &= \frac{1}{u(t)}\{u; x\}^{\nabla}(t) \\
&= \frac{1}{u(t)}\{pW(u, x)\}^{\nabla}(t) \\
&= \frac{1}{u(t)}\left\{puu^{\sigma}\left[\frac{x}{u}\right]^{\Delta}\right\}^{\nabla}(t) \\
&= \alpha_1(t)\{\alpha_2[\alpha_1 x]^{\Delta}\}^{\nabla}(t),
\end{aligned}
$$

for $t \in \mathcal{I}$, where α_1 and α_2 are as described in the theorem.   □

Example 1.51. Find two Polya factorization for the following dynamic equation.

$$x^{\Delta\nabla} - 7x^{\nabla} + 10x^{\rho} = 0.$$

The characteristic equation is

$$\lambda^2 - 7\lambda + 10 = 0,$$

which has roots $\lambda_1 = 2$ and $\lambda_2 = 5$. Thus $e_2(t, t_0)$ and $e_5(t, t_0)$ are positive solutions of this dynamic equation. For the first Polya factorization, let $u(t) = e_2(t, t_0)$. Then

$$\alpha_1(t) = \frac{1}{e_2(t, t_0)} = e_{\ominus 2}(t, t_0), \quad \text{and} \quad \alpha_2(t) = \hat{e}_{(-7-10\nu(t))}(t, t_0)e_2(t, t_0)e_2^{\sigma}(t, t_0).$$

We can use properties of exponential functions to simplify $\alpha_2(t)$. We get

$$\alpha_2(t) = e_2^\sigma(t, t_0)e_{\ominus 5}(t, t_0),$$

and we see that

$$Lx = e_{\ominus 2}(t, t_0)\{e_2^\sigma(t, t_0)e_{\ominus 5}(t, t_0)[e_{\ominus 2}(t, t_0)x]^\Delta\}^\nabla.$$

For the second Polya factorization, we will let $u(t) = e_5(t, t_0)$. Then

$$\alpha_1(t) = \frac{1}{e_5(t, t_0)} = e_{\ominus 5}(t, t_0), \quad \text{and} \quad \alpha_2(t) = \hat{e}_{(-7-10\nu(t))}(t, t_0)e_5(t, t_0)e_5^\sigma(t, t_0).$$

Again, we use properties of exponential functions to simplify $\alpha_2(t)$, giving

$$\alpha_2(t) = e_5^\sigma(t, t_0)e_{\ominus 2}(t, t_0),$$

and thus

$$Lx = e_{\ominus 5}(t, t_0)\{e_5^\sigma(t, t_0)e_{\ominus 2}(t, t_0)[e_{\ominus 5}(t, t_0)x]^\Delta\}^\nabla.$$

Exercise 1.52. Find two Polya factorizations for each of the following dynamic equations.

   (i) $x^{\Delta\nabla} - 8x^\nabla + 12x^\rho = 0$;

   (ii) $x^{\Delta\nabla} - 4x^\nabla + 3x^\rho = 0$;

   (iii) $x^{\Delta\nabla} - 3x^\nabla + 2x^\rho = 0$.

Based on the previous example, and exercise, it is clear that Polya factorizations are not unique. There is a similar factorization, called a *Trench factorization* which is essentially unique. The difference between a Polya factorization and a Trench factorization is whether or not a particular integral diverges.

Theorem 1.53 (Trench Factorization). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, assume $\rho(\omega) = \omega$. If* (1.1) *is nonoscillatory on $[a, \omega)$, then there is $t_0 \in \mathbb{T}$ such that for any $x \in \mathbb{D}$, we get the* Trench Factorization

$$Lx(t) = \beta_1(t)\{\beta_2[\beta_1 x]^\Delta\}^\nabla(t)$$

for $t \in [t_0, \omega)$, where $\beta_1, \beta_2 > 0$ on $[t_0, \omega)$, and

$$\int_{t_0}^\omega \frac{1}{\beta_2(t)}\Delta t = \infty.$$

Proof. Since (1.1) is nonoscillatory on $[a, \omega)$, (1.1) has a positive solution, u on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$. Then by Theorem 1.50, Lx has a Polya factorization on $[t_0, \omega)$. Thus there are functions α_1 and α_2 such that

$$Lx(t) = \alpha_1(t)\{\alpha_2[\alpha_1 x]^\Delta\}^\nabla(t) \text{ for } t \in [t_0, \omega),$$

with

$$\alpha_1 = \frac{1}{u} \text{ and } \alpha_2 = puu^\sigma.$$

Now, if

$$\int_{t_0}^{\omega} \frac{1}{\alpha_2(t)} \Delta t = \infty,$$

then take $\beta_1(t) = \alpha_1(t)$, and $\beta_2(t) = \alpha_2(t)$, and we are done. Therefore, assume that

$$\int_{t_0}^{\omega} \frac{1}{\alpha_2(t)} \Delta t < \infty.$$

In this case, let

$$\beta_1(t) = \frac{\alpha_1(t)}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \text{ and } \beta_2(t) = \alpha_2(t) \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s$$

for $t \in [t_0, \omega)$. Note that as $\alpha_1, \alpha_2 > 0$, we have $\beta_1, \beta_2 > 0$ as well. Also,

$$
\begin{aligned}
\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t &= \lim_{b \to \omega, b \in \mathbb{T}} \int_{t_0}^{b} \frac{1}{\alpha_2(t) \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \Delta t \\
&= \lim_{b \to \omega, b \in \mathbb{T}} \int_{t_0}^{b} \frac{\frac{1}{\alpha_2(s)}}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \Delta t \\
&= \lim_{b \to \omega, b \in \mathbb{T}} \int_{t_0}^{b} \left[\frac{1}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \right]^{\Delta} \Delta t \\
&= \lim_{b \to \omega, b \in \mathbb{T}} \left[\frac{1}{\int_b^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \right] \\
&= \infty.
\end{aligned}
$$

Now let $x \in \mathbb{D}$. Then

$$[\beta_1 x]^{\Delta}(t) = \left[\frac{\alpha_1(t) x(t)}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s} \right]^{\Delta} = \frac{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s [\alpha_1(t) x(t)]^{\Delta} + \alpha_1(t) x(t) \frac{1}{\alpha_2(t)}}{\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \int_{\sigma(t)}^{\omega} \frac{1}{\alpha_2(s)} \Delta s}$$

for $t \in [t_0, \omega)$. So we get

$$\beta_2(t) [\beta_1(t) x]^{\Delta} = \alpha_2(t) [\alpha_1(t) x(t)]^{\Delta} \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s + \alpha_1(t) x(t)$$

for $t \in [t_0, \omega)$. Taking the ∇-derivative of both sides gives

$$
\begin{aligned}
\left\{ \beta_2(t) [\beta_1(t) x(t)]^{\Delta} \right\}^{\nabla} &= \left\{ \alpha_2(t) [\alpha_1(t) x(t)]^{\Delta} \right\}^{\nabla} \int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \\
&\quad + \left\{ \alpha_2(t) [\alpha_1(t) x(t)]^{\Delta} \right\}^{\rho} \left[\int_t^{\omega} \frac{1}{\alpha_2(s)} \Delta s \right]^{\nabla} \\
&\quad + [\alpha_1(t) x(t)]^{\nabla}
\end{aligned}
$$

for $t \in [t_0, \omega)$. We now claim that the last two terms in this expression cancel. Looking only at these last two terms, put the expression back in terms of our positive solution u. We get

$$\left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\rho \left[\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s\right]^\nabla + [\alpha_1(t)x(t)]^\nabla$$

$$= [p(t)u(t)u^\sigma(t)]^\rho \left[\frac{x(t)}{u(t)}\right]^{\Delta\rho} \left[\int_t^\omega \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla + \left[\frac{x(t)}{u(t)}\right]^\nabla.$$

Now consider two cases:

Case 1: $t \in \mathbb{T}_A$. Then Theorem 1.8 applies, and we get

$$[p(t)u(t)u^\sigma(t)]^\rho \left[\frac{x(t)}{u(t)}\right]^{\Delta\rho} \left[\int_t^\omega \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla + \left[\frac{x(t)}{u(t)}\right]^\nabla$$

$$= -\frac{p^\rho(t)u^\rho(t)u(t)\left[\frac{x(t)}{u(t)}\right]^\nabla}{p^\rho(t)u^\rho(t)u(t)} + \left[\frac{x(t)}{u(t)}\right]^\nabla$$

$$= -\left[\frac{x(t)}{u(t)}\right]^\nabla + \left[\frac{x(t)}{u(t)}\right]^\nabla$$

$$= 0.$$

Case 2: $t \in A$. In this case we have that $\rho(t) = t$, and we get

$$[p(t)u(t)u^\sigma(t)]^\rho \left[\frac{x(t)}{u(t)}\right]^{\Delta\rho} \left[\int_t^\omega \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s\right]^\nabla + \left[\frac{x(t)}{u(t)}\right]^\nabla$$

$$= -\frac{[p(t)u(t)u^\sigma(t)]\left[\frac{u(t)x^\Delta(t)-x(t)u^\Delta(t)}{u(t)u^\sigma(t)}\right]}{p(t)u^2(t)} + \frac{u(t)x^\nabla(t) - x(t)u^\nabla(t)}{u^2(t)}$$

$$= -\frac{u(t)x^{\Delta\rho}(t) - x(t)u^{\Delta\rho}(t)}{u^2(t)} + \frac{u(t)x^\nabla(t) - x(t)u^\nabla(t)}{u^2(t)}$$

$$= \frac{-u(t)x^\nabla(t) + x(t)u^\nabla(t) + u(t)x^\nabla(t) - x(t)u^\nabla(t)}{u^2(t)}$$

$$= 0.$$

Here, we have made use of the fact that $x, u \in \mathbb{D}$, which gives us that $x^{\Delta\rho} = x^\nabla$ and $u^{\Delta\rho} = u^\nabla$.

In either case, the last two terms cancel, and we have that

$$\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s.$$

It then follows that

$$\beta_1(t)\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \alpha_1(t)\left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla = Lx(t),$$

for $t \in [t_0, \omega)$ and the proof is complete. $\square$

Example 1.54. Find a Trench factorization for the dynamic equation

$$x^{\Delta\nabla} - 7x^{\nabla} + 10x^{\rho} = 0,$$

assuming that $\sup \mathbb{T} = \infty$. From an earlier exercise, we have that both

$$Lx = e_{\ominus 2}(t, t_0)\{e_2^\sigma(t, t_0)e_{\ominus 5}(t, t_0)[e_{\ominus 2}(t, t_0)x]^{\Delta}\}^{\nabla},$$

and

$$Lx = e_{\ominus 5}(t, t_0)\{e_5^\sigma(t, t_0)e_{\ominus 2}(t, t_0)[e_{\ominus 5}(t, t_0)x]^{\Delta}\}^{\nabla}.$$

are Polya factorizations for this dynamic equation. Looking at the first one we see that

$$
\begin{aligned}
\int_{t_0}^{\infty} \frac{1}{\beta_2(t)}\Delta t &= \int_{t_0}^{\infty} \frac{1}{e_2^\sigma(t, t_0)e_{\ominus 5}(t, t_0)}\Delta t \\
&= \lim_{b\to\infty} \int_{t_0}^{b} \frac{e_5(t, t_0)}{e_2^\sigma(t, t_0)}\Delta t \\
&= \lim_{b\to\infty} \int_{t_0}^{b} \frac{1}{3}(e_{5\ominus 2}^{\Delta}(t, t_0))\Delta t \\
&= \lim_{b\to\infty} \frac{1}{3}\left[e_{5\ominus 2}(b, t_0) - e_{5\ominus 2}(t_0, t_0)\right] \\
&= \lim_{b\to\infty} \left[\frac{e_5(b, t_0)}{3e_2(b, t_0)} - \frac{1}{3}\right] \\
&= \infty.
\end{aligned}
$$

Whereas, looking at the second one, we get

$$
\begin{aligned}
\int_{t_0}^{\infty} \frac{1}{\beta_2(t)}\Delta t &= \int_{t_0}^{\infty} \frac{1}{e_5^\sigma(t, t_0)e_{\ominus 2}(t, t_0)}\Delta t \\
&= \lim_{b\to\infty} \int_{t_0}^{b} \frac{e_2(t, t_0)}{e_5^\sigma(t, t_0)}\Delta t \\
&= \lim_{b\to\infty} \int_{t_0}^{b} -\frac{1}{3}(e_{2\ominus 5}^{\Delta}(t, t_0))\Delta t \\
&= \lim_{b\to\infty} -\frac{1}{3}\left[e_{2\ominus 5}(b, t_0) - e_{2\ominus 5}(t_0, t_0)\right] \\
&= \lim_{b\to\infty} \left[-\frac{e_2(b, t_0)}{3e_5(b, t_0)} + \frac{1}{3}\right] \\
&= \frac{1}{3}.
\end{aligned}
$$

So, the first factorization is a Trench factorization, while the second one is not.

Exercise 1.55. Find a Trench factorization for each of the dynamic equations in Exercise 1.52.

The existence of a Trench factorization allows us to prove the following theorem. Notice that the divergence of the integral in the Trench factorization plays a key role here.

Theorem 1.56 (Recessive and Dominant Solutions). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, then we assume $\rho(\omega) = \omega$. If (1.1) is nonoscillatory on $[a, \omega)$, then there is a solution, u, called a recessive solution at ω, such that u is positive on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$, and if v is any second, linearly independent solution, called a dominant solution at ω, the following hold.*

(i) $\lim_{t \to \omega^-} \frac{u(t)}{v(t)} = 0$,

(ii) $\int_{t_0}^{\omega} \frac{1}{p(t)u(t)u^{\sigma}(t)} \Delta t = \infty$,

(iii) $\int_{b}^{\omega} \frac{1}{p(t)v(t)v^{\sigma}(t)} \Delta t < \infty$ for $b < \omega$, sufficiently close, and

(iv) $\frac{p(t)v^{\Delta}(t)}{v(t)} > \frac{p(t)u^{\Delta}(t)}{u(t)}$ for $t < \omega$, sufficiently close.

The recessive solution, u, is unique, up to multiplication by a nonzero constant.

Proof. As (1.1) is nonoscillatory, by Theorem 1.53, there is a Trench Factorization:
$$Lx(t) = \beta_1(t)\{\beta_2[\beta_1 x]^{\Delta}\}^{\nabla}(t),$$
where $\beta_1, \beta_2 > 0$ on $[t_0, \omega)$, and
$$\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t = \infty.$$

Then if $u(t) = \frac{1}{\beta_1(t)}$, u is a positive solution of (1.1). Now, let
$$v_0(t) = \frac{1}{\beta_1(t)} \int_{t_0}^{\omega} \frac{1}{\beta_2(s)} \Delta s.$$

Then,
$$
\begin{aligned}
Lv_0(t) &= \beta_1(t)\{\beta_2[\beta_1 v_0]^{\Delta}\}^{\nabla}(t) \\
&= \beta_1(t)\{\beta_2[\beta_1 \frac{1}{\beta_1} \int_{t_0}^{\omega} \frac{1}{\beta_2(s)} \Delta s]^{\Delta}\}^{\nabla}(t) \\
&\doteq \beta_1(t)\{\beta_2[\int_{t_0}^{\omega} \frac{1}{\beta_2(s)} \Delta s]^{\Delta}\}^{\nabla}(t) \\
&= \beta_1(t)\{\beta_2(t)\frac{1}{\beta_2(t)}\}^{\nabla} \\
&= \beta_1(t)\{1\}^{\nabla} \\
&= 0.
\end{aligned}
$$

So v_0 is a solution of (1.1). Note that

$$\lim_{t \to \omega-} \frac{u(t)}{v_0(t)} = \lim_{t_0 \to \omega} \frac{1}{\int_{t_0}^{\omega} \frac{1}{\beta_2(s)} \Delta s} = 0,$$

as $\int_{t_0}^{\omega} \frac{1}{\beta_2(t)} \Delta t = \infty$. Now

$$\left(\frac{v_0}{u}\right)^{\Delta}(t) = \frac{W(u, v_0)(t)}{u(t)u^{\sigma}(t)} = \frac{C}{p(t)u(t)u^{\sigma}(t)},$$

where C is a constant by Theorem 1.35. Note that $C \neq 0$, since u and v_0 are linearly independent. Integrating both sides of this last equation from t_0 to t, we get

$$\frac{v_0(t)}{u(t)} = \int_{t_0}^{t} \frac{C}{p(s)u(s)u^{\sigma}(s)} \Delta s.$$

Taking the limit as $t \to \omega$, we get

$$\lim_{t \to \omega} \frac{v_0(t)}{u(t)} = \int_{t_0}^{\omega} \frac{C}{p(s)u(s)u^{\sigma}(s)} \Delta s,$$

and we see that

$$\int_{t_0}^{\omega} \frac{C}{p(s)u(s)u^{\sigma}(s)} \Delta s = \infty,$$

as desired.

Now let v be any solution of (1.1) such that u and v are linearly independent. Then

$$v(t) = c_1 u(t) + c_2 v_0(t), \quad \text{where } c_2 \neq 0,$$

and

$$\begin{aligned}
\lim_{t \to \omega-} \frac{u(t)}{v(t)} &= \lim_{t \to \omega-} \frac{u(t)}{c_1 u(t) + c_2 v_0(t)} \\
&= \lim_{t \to \omega-} \frac{\frac{u(t)}{v_0(t)}}{c_1 \frac{u(t)}{v_0(t)} + c_2} \\
&= 0.
\end{aligned}$$

Now, let v be a fixed solution of (1.1) such that u and v are linearly independent. Choose $t_1 \in [t_0, \omega)$ such that $v(t)v^{\sigma}(t) > 0$ on $[t_1, \omega)$. Then for $t \in [t_1, \omega)$,

$$\left(\frac{u}{v}\right)^{\Delta}(t) = \left(\frac{vu^{\Delta} - uv^{\Delta}}{vv^{\sigma}}\right)(t) = \frac{W(v, u)(t)}{v(t)v^{\sigma}(t)} = \frac{C_1}{p(t)v(t)v^{\sigma}(t)}, \quad \text{where } C_1 \neq 0.$$

Integrating,

$$\frac{u(t)}{v(t)} - \frac{u(t_1)}{v(t_1)} = \int_{t_1}^{t} \frac{C_1}{p(s)v(s)v^{\sigma}(s)} \Delta s.$$

Letting $t \to \omega-$, we see that

$$-\frac{u(t_1)}{v(t_1)} = \int_{t_1}^{\omega} \frac{C_1}{p(s)v(s)v^{\sigma}(s)} \Delta s,$$

which implies that

$$\int_{t_1}^{\omega} \frac{1}{p(s)v(s)v^{\sigma}(s)} \Delta s < \infty.$$

Furthermore, for $t \in [t_1, \omega)$,

$$\frac{p(t)v^{\Delta}(t)}{v(t)} - \frac{p(t)u^{\Delta}(t)}{u(t)} = \frac{p(t)W(u,v)(t)}{u(t)v(t)} = \frac{C_2}{u(t)v(t)}, \quad \text{where } C_2 \neq 0.$$

It remains to show that $C_2 > 0$. We have

$$\lim_{t \to \omega} \frac{v(t)}{u(t)} = \infty,$$

and

$$\left(\frac{v}{u}\right)^{\Delta}(t) = \frac{W(u,v)(t)}{u(t)u^{\sigma}(t)} = \frac{C_2}{p(t)u(t)u^{\sigma}(t)},$$

which implies that $C_2 > 0$, as desired.

Finally, we need to establish uniqueness, up to multiplication by a nonzero constant. Let u_1 be a recessive solution of (1.1), and suppose u_2 is another recessive solution. If u_1 and u_2 were linearly independent, u_2 would be a dominant solution. Hence u_1 and u_2 must be linearly dependent, and we see that $u_2 = ku_2$ for some nonzero constant k.   □

Example 1.57. Find recessive and dominant solutions for the dynamic equation

$$x^{\Delta\nabla} - 7x^{\nabla} + 10x^{\rho} = 0,$$

assuming that $\sup \mathbb{T} = \infty$, and verify that the properties indicated in Theorem 1.56 hold.

By previous work, we know that $e_2(t, t_0)$ and $e_5(t, t_0)$ are linearly independent solutions of this dynamic equation. I now claim that $u(t) := e_2(t, t_0)$ is the recessive solution, and $v(t) := e_5(t, t_0)$ is a dominant solution. To verify this, we need to show four things:

First, we must show that

$$\lim_{t \to \infty} \frac{u(t)}{v(t)} = \lim_{t \to \infty} \frac{e_2(t, t_0)}{e_5(t, t_0)} = 0.$$

This is clear by inspection. Second, we must show that

$$\int_{t_0}^{\infty} \frac{1}{p(t)u(t)u^{\sigma}(t)} \Delta t = \infty.$$

Simplifying this expression, we get

$$
\begin{aligned}
\int_{t_0}^{\infty} \frac{1}{p(t)u(t)u^{\sigma}(t)} \Delta t
&= \int_{t_0}^{\infty} \frac{1}{\hat{e}_{(-7-10\nu(t))}(t,t_0) e_2(t,t_0) e_2^{\sigma}(t,t_0)} \Delta t \\
&= \lim_{b \to \infty} \int_{t_0}^{b} \frac{e_5(t,t_0)}{e_2^{\sigma}(t,t_0)} \Delta t \\
&= \lim_{b \to \infty} \int_{t_0}^{b} \frac{1}{3}(e_{5\ominus2}^{\Delta}(t,t_0)) \Delta t \\
&= \lim_{b \to \infty} \frac{1}{3} \left[e_{5\ominus2}(b,t_0) - e_{5\ominus2}(t_0,t_0) \right] \\
&= \lim_{b \to \infty} \left[\frac{e_5(b,t_0)}{3e_2(b,t_0)} - \frac{1}{3} \right] \\
&= \infty.
\end{aligned}
$$

The third thing we must verify is that

$$
\int_{t_0}^{\infty} \frac{1}{p(t)v(t)v^{\sigma}(t)} \Delta t < \infty.
$$

Simplifying this expression, we get

$$
\begin{aligned}
\int_{t_0}^{\infty} \frac{1}{p(t)v(t)v^{\sigma}(t)} \Delta t
&= \int_{t_0}^{\infty} \frac{1}{\hat{e}_{(-7-10\nu(t))}(t,t_0) e_5(t,t_0) e_5^{\sigma}(t,t_0)} \Delta t \\
&= \lim_{b \to \infty} \int_{t_0}^{b} \frac{e_2(t,t_0)}{e_5^{\sigma}(t,t_0)} \Delta t \\
&= \lim_{b \to \infty} \int_{t_0}^{b} -\frac{1}{3}(e_{2\ominus5}^{\Delta}(t,t_0)) \Delta t \\
&= \lim_{b \to \infty} -\frac{1}{3} \left[e_{2\ominus5}(b,t_0) - e_{2\ominus5}(t_0,t_0) \right] \\
&= \lim_{b \to \infty} \left[-\frac{e_2(b,t_0)}{3e_5(b,t_0)} + \frac{1}{3} \right] \\
&= \frac{1}{3} < \infty.
\end{aligned}
$$

Now, finally, we must verify that

$$
\frac{p(t)v^{\Delta}(t)}{v(t)} > \frac{p(t)u^{\Delta}(t)}{u(t)} \quad \text{for } t \text{ sufficiently large,}
$$

or

$$
\frac{\hat{e}_{(-7-10\nu(t))}(t,t_0) e_5^{\Delta}(t,t_0)}{e_5(t,t_0)} > \frac{\hat{e}_{(-7-10\nu(t))}(t,t_0) e_2^{\Delta}(t,t_0)}{e_2(t,t_0)} \quad \text{for } t \text{ sufficiently large.}
$$

Looking at the first expression, we get

$$\frac{\hat{e}_{(-7-10\nu(t))}(t,t_0)e_5^\Delta(t,t_0)}{e_5(t,t_0)} = \frac{e_{\ominus 2}(t,t_0)e_{\ominus 5}(t,t_0)5e_5(t,t_0)}{e_5(t,t_0)}$$

$$= \frac{5}{e_2(t,t_0)e_5(t,t_0)}.$$

The second expression gives

$$\frac{\hat{e}_{(-7-10\nu(t))}(t,t_0)e_2^\Delta(t,t_0)}{e_2(t,t_0)} = \frac{e_{\ominus 2}(t,t_0)e_{\ominus 5}(t,t_0)2e_2(t,t_0)}{e_2(t,t_0)}$$

$$= \frac{2}{e_2(t,t_0)e_5(t,t_0)}.$$

As both of the exponential functions $e_2(t,t_0)$ and $e_5(t,t_0)$ are positive, we see that

$$\frac{5}{e_2(t,t_0)e_5(t,t_0)} > \frac{2}{e_2(t,t_0)e_5(t,t_0)}$$

holds for all $t \in \mathbb{T}$. Thus $u(t) = e_2(t,t_0)$ is the recessive solution, and $v(t) = e_5(t,t_0)$ is a dominant solution as we claimed.

Exercise 1.58. For each of the dynamic equations in Exercise 1.52, find recessive and dominant solutions and verify that the properties indicated in Theorem 1.56 hold.

6   The Riccati Equation

Usually, linear dynamic equations are considerably easier to solve than nonlinear ones. In this section, we are going to discuss the relationship between a particular nonlinear equation, called the Riccati equation, and our self-adjoint equation. We will see that there is a correspondence between solutions of these two equations. The Riccati equation is defined by

$$Rz = 0, \quad \text{where } Rz(t) := z^\nabla(t) + q(t) + \frac{(z^\rho(t))^2}{p^\rho(t) + \nu(t)z^\rho(t)} \tag{1.5}$$

for $t \in \mathbb{T}_\kappa^\kappa$. Here we assume that $p : \mathbb{T} \to \mathbb{R}$ is continuous, $q : \mathbb{T} \to \mathbb{R}$ is ld-continuous and that

$$p(t) > 0 \text{ for all } t \in \mathbb{T}.$$

Define the set $\mathbb{D}_R$ to be the set of all functions $z : \mathbb{T}^\kappa \to \mathbb{R}$ such that $z^\nabla : \mathbb{T}_\kappa^\kappa \to \mathbb{R}$ is ld-continuous and such that $p^\rho(t) + \nu(t)z^\rho(t) > 0$ for any $t \in \mathbb{T}_\kappa^\kappa$. A function $z \in \mathbb{D}_R$ is said to be a solution of $Rz = 0$ on $\mathbb{T}^\kappa$ provided $Rz(t) = 0$ for all $t \in \mathbb{T}_\kappa^\kappa$.

Then we have the following theorem:

Theorem 1.59. *Assume $x \in \mathbb{D}$ has no generalized zeros in $\mathbb{T}$, and z is defined by the Riccati substitution*

$$z(t) = \frac{p(t)x^\Delta(t)}{x(t)}, \qquad (1.6)$$

for $t \in \mathbb{T}^\kappa$. Then $z \in \mathbb{D}_R$, and

$$Lx(t) = x(t)Rz(t)$$

for $t \in \mathbb{T}^\kappa_\kappa$.

Proof. We first wish to show that $z \in \mathbb{D}_R$. We have by the quotient rule

$$z^\nabla(t) = \left[\frac{p(t)x^\Delta(t)}{x(t)}\right]^\nabla = \frac{x(t)[p(t)x^\Delta(t)]^\nabla - p(t)x^\Delta(t)x^\nabla(t)}{x(t)x^\rho(t)},$$

which is ld-continuous on $\mathbb{T}^\kappa_\kappa$, since $x \in \mathbb{D}$. Next, note that

$$
\begin{aligned}
p^\rho(t) + \nu(t)z^\rho(t) &= p^\rho(t) + \nu(t)\frac{p^\rho(t)x^{\Delta\rho}(t)}{x^\rho(t)} \\
&= \frac{p^\rho(t)(x^\rho(t) + \nu(t)x^\nabla(t))}{x^\rho(t)} \\
&= \frac{p^\rho(t)x(t)}{x^\rho(t)} > 0
\end{aligned}
$$

for all $t \in \mathbb{T}^\kappa_\kappa$, since x has no generalized zeros in $\mathbb{T}$. It remains to show that $x(t)Rz(t) = Lx(t)$ for $t \in \mathbb{T}^\kappa_\kappa$. Suppressing the arguments, we get

$$
\begin{aligned}
xRz &= x\left[z^\nabla + q + \frac{(z^\rho)^2}{p^\rho + \nu z^\rho}\right] \\
&= x\left[\left(\frac{px^\Delta}{x}\right)^\nabla + q + \frac{1}{p^\rho + \nu\left(\frac{px^\Delta}{x}\right)^\rho}\left(\left(\frac{px^\Delta}{x}\right)^\rho\right)^2\right] \\
&= x\left[\frac{x(px^\Delta)^\nabla - px^\Delta x^\nabla}{xx^\rho} + q + \frac{x^\rho}{p^\rho(x^\rho + \nu x^{\Delta\rho})}\frac{(p^\rho)^2(x^{\Delta\rho})^2}{(x^\rho)^2}\right] \\
&= (px^\Delta)^\nabla\frac{x}{x^\rho} - \frac{px^\Delta x^\nabla}{x^\rho} + qx + \frac{xp^\rho(x^\nabla)^2}{x^\rho(x^\rho + \nu x^\nabla)} \\
&= (px^\Delta)^\nabla\left(1 + \frac{\nu x^\nabla}{x^\rho}\right) - \frac{px^\Delta x^\nabla}{x^\rho} + qx + \frac{xp^\rho(x^\nabla)^2}{x^\rho x} \\
&= (px^\Delta)^\nabla + qx + \frac{(px^\Delta)^\nabla \nu x^\nabla}{x^\rho} - \frac{px^\Delta x^\nabla}{x^\rho} + \frac{p^\rho(x^\nabla)^2}{x^\rho} \\
&= Lx + \frac{(px^\Delta)^\nabla \nu x^\nabla - px^\Delta x^\nabla + p^\rho(x^\nabla)(x^{\Delta\rho})}{x^\rho} \\
&= Lx + \frac{x^\nabla(p^\rho x^{\Delta\rho} + \nu(px^\Delta)^\nabla) - px^\Delta x^\nabla}{x^\rho}
\end{aligned}
$$

$$\begin{aligned} &= Lx + \frac{x^\nabla(px^\Delta) - px^\Delta x^\nabla}{x^\rho} \\ &= Lx. \end{aligned}$$

Again, we have made use of the fact that $x \in \mathbb{D}$ and applied Theorem 1.8. $\square$

Theorem 1.60. *The self-adjoint equation* (1.1) *has a positive solution on* $\mathbb{T}$ *if and only if the Riccati equation* (1.5) *has a solution* z *on* $\mathbb{T}^\kappa$.

Proof. First, assume that x is a positive solution of (1.1), and let z be defined by the Riccati substitution (1.6). Then by Theorem 1.59, $z \in \mathbb{D}_R$, and $Lx = xRz$. Since x is a solution of $Lx = 0$ and has no generalized zeros, it follows that $Rz = 0$, as desired.

Conversely, assume that z is a solution of the Riccati equation, (1.5) on $\mathbb{T}^\kappa$. Then $z \in \mathbb{D}_R$, so $p^\rho(t) + \nu(t)z^\rho(t) > 0$ for all $t \in \mathbb{T}^\kappa_\kappa$, and z is continuous on $\mathbb{T}^\kappa$. This gives us that $-\left(\frac{z}{p}\right)^\rho \in \mathcal{R}^+_\nu$, and thus, by Lemma 1.15, $\frac{z}{p} \in \mathcal{R}^+$. Now, let $t_0 \in \mathbb{T}$, and let x be the solution of the initial value problem

$$x^\Delta = \frac{z(t)}{p(t)}x, \quad x(t_0) = 1.$$

Note that although $\frac{z}{p}$ is only defined on $\mathbb{T}^\kappa$, x is defined on $\mathbb{T}$. Furthermore, as $x(t) = e_{\frac{z}{p}}(t, t_0)$, x is continuous and positive on $\mathbb{T}$. Next, consider

$$\begin{aligned} \left[p(t)x^\Delta(t)\right]^\nabla &= \left[z(t)x(t)\right]^\nabla \\ &= z^\nabla(t)x^\rho(t) + z(t)x^\nabla(t) \\ &= z^\nabla(t)x^\rho(t) + z(t)x^{\Delta\rho}(t), \end{aligned}$$

which is ld-continuous on $\mathbb{T}^\kappa_\kappa$. Hence $x \in \mathbb{D}$. Moreover, we see that

$$z(t) = \frac{p(t)x^\Delta(t)}{x(t)},$$

so by Theorem 1.59 $Lx = xRz = 0$. Hence x is the desired positive solution of (1.1). $\square$

The preceding theorem allows us to find solutions of the Riccati equation by solving the corresponding self-adjoint equation.

Example 1.61. Solve the Riccati equation

$$z^\nabla + 12\hat{e}_{(-7-12\nu)}(t, t_0) + \frac{(z^\rho)^2}{\hat{e}^\rho_{(-7-12\nu)}(t, t_0) + \nu(t)z^\rho} = 0.$$

The associated self-adjoint equation is

$$\left[\hat{e}_{(-7-12\nu)}(t, t_0)x^\Delta\right]^\nabla + 12\hat{e}_{(-7-12\nu)}(t, t_0)x = 0.$$

Rewriting this as a second-order linear equation, we get

$$x^{\Delta\nabla} - 7x^{\nabla} + 12x^{\rho} = 0.$$

The characteristic equation, then, is

$$\lambda^2 - 7\lambda + 12 = 0,$$

which has roots $\lambda_1 = 3$ and $\lambda_2 = 4$. Thus $e_3(t, t_0)$ and $e_4(t, t_0)$ are positive solutions of this dynamic equation. So, let $x(t) = e_3(t, t_0)$, and let

$$z(t) := \frac{p(t)x^{\Delta}(t)}{x(t)} = \frac{\hat{e}_{(-7-12\nu)}(t, t_0)3e_3(t, t_0)}{e_3(t, t_0)} = 3\hat{e}_{(-7-12\nu)}(t, t_0).$$

By Theorem 1.60, z is a solution of our Riccati equation. We will verify this by direct calculation. Recall that we can simplify z to get $z(t) = 3\hat{e}_{-3}(t, t_0)\hat{e}_{-4}(t, t_0)$. To simplify the notation, we are going to drop the argument, t, from our functions. Then, substitution gives

$$
\begin{aligned}
Rz &= z^{\nabla} + 12\hat{e}_{(-7-12\nu)} + \frac{(z^{\rho})^2}{\hat{e}^{\rho}_{(-7-12\nu)} + \nu z^{\rho}} \\
&= (3\hat{e}_{-3}\hat{e}_{-4})^{\nabla} + 12\hat{e}_{-3}\hat{e}_{-4} + \frac{9(\hat{e}^{\rho}_{-3})^2(\hat{e}^{\rho}_{-4})^2}{\hat{e}^{\rho}_{-3}\hat{e}^{\rho}_{-4} + 3\nu\hat{e}^{\rho}_{-3}\hat{e}^{\rho}_{-4}} \\
&= -9\hat{e}_{-3}\hat{e}_{-4} - 12\hat{e}^{\rho}_{-3}\hat{e}_{-4} + 12\hat{e}_{-3}\hat{e}_{-4} + \frac{9\hat{e}^{\rho}_{-3}\hat{e}^{\rho}_{-4}}{1 + 3\nu} \\
&= -9\hat{e}_{-3}\hat{e}_{-4} - 12\hat{e}_{-4}\left[\hat{e}_{-3} + 3\nu\hat{e}_{-3}\right] + 12\hat{e}_{-3}\hat{e}_{-4} + \frac{9\hat{e}^{\rho}_{-3}\hat{e}^{\rho}_{-4}}{1 + 3\nu} \\
&= -9\hat{e}_{-3}\hat{e}_{-4} - 36\nu\hat{e}_{-3}\hat{e}_{-4} + \frac{9\hat{e}^{\rho}_{-3}\hat{e}^{\rho}_{-4}}{1 + 3\nu}.
\end{aligned}
$$

Now, note that

$$
\begin{aligned}
\hat{e}^{\rho}_{-3}\hat{e}^{\rho}_{-4} &= \left[\hat{e}_{-3} + 3\nu\hat{e}_{-3}\right]\left[\hat{e}_{-4} + 4\nu\hat{e}_{-4}\right] \\
&= \hat{e}_{-3}\hat{e}_{-4} + 7\nu\hat{e}_{-3}\hat{e}_{-4} + 12\nu^2\hat{e}_{-3}\hat{e}_{-4},
\end{aligned}
$$

so we get

$$
\begin{aligned}
Rz &= \frac{1}{1 + \nu}\left[-9\hat{e}_{-3}\hat{e}_{-4} - 36\nu\hat{e}_{-3}\hat{e}_{-4}\right](1 + 3\nu) \\
&\qquad\qquad + 9\left[\hat{e}_{-3}\hat{e}_{-4} + 7\nu\hat{e}_{-3}\hat{e}_{-4} + 12\nu^2\hat{e}_{-3}\hat{e}_{-4}\right] \\
&= \frac{9}{1 + \nu}\left[(-\hat{e}_{-3}\hat{e}_{-4} - 4\nu\hat{e}_{-3}\hat{e}_{-4})(1 + 3\nu)\right. \\
&\qquad\qquad \left. + \hat{e}_{-3}\hat{e}_{-4} + 7\nu\hat{e}_{-3}\hat{e}_{-4} + 12\nu^2\hat{e}_{-3}\hat{e}_{-4}\right] \\
&= \frac{9}{1 + \nu}\left[-\hat{e}_{-3}\hat{e}_{-4} - 4\nu\hat{e}_{-3}\hat{e}_{-4} - 3\nu\hat{e}_{-3}\hat{e}_{-4} - 12\nu^2\hat{e}_{-3}\hat{e}_{-4}\right. \\
&\qquad\qquad \left. + \hat{e}_{-3}\hat{e}_{-4} + 7\nu\hat{e}_{-3}\hat{e}_{-4} + 12\nu^2\hat{e}_{-3}\hat{e}_{-4}\right] \\
&= 0.
\end{aligned}
$$

Exercise 1.62. Solve the following Riccati equations:

(i) $z^\nabla + 3\hat{e}_{(-4-3\nu)}(t, t_0) + \dfrac{(z^\rho)^2}{\hat{e}^\rho_{(-4-3\nu)}(t,t_0) + \nu(t)z^\rho} = 0;$

(ii) $z^\nabla + 10\hat{e}_{(-7-10\nu)}(t, t_0) + \dfrac{(z^\rho)^2}{\hat{e}^\rho_{(-7-10\nu)}(t,t_0) + \nu(t)z^\rho} = 0;$

(iii) $z^\nabla + 12\hat{e}_{(-8-12\nu)}(t, t_0) + \dfrac{(z^\rho)^2}{\hat{e}^\rho_{(-8-12\nu)}(t,t_0) + \nu(t)z^\rho} = 0.$

Now, define $\mathbb{A}$ to be the set of functions

$$\mathbb{A} := \{u \in C^1_{pld}([\rho(a), \sigma(b)], \mathbb{R}) : u(\rho(a)) = u(\sigma(b)) = 0\}.$$

Here, C^1_{pld} denotes the set of all continuous functions whose ∇-derivatives are piecewise ld-continuous. Then we define the quadratic functional $\mathcal{F}$ on $\mathbb{A}$, by

$$\mathcal{F}(u) := \int_{\rho(a)}^{\sigma(b)} \left[p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t)\right] \nabla t.$$

Definition 1.63. We say $\mathcal{F}$ is *positive definite* on $\mathbb{A}$ provided $\mathcal{F}(u) \geq 0$ for all $u \in \mathbb{A}$, and $\mathcal{F}(u) = 0$ if and only if $u = 0$.

Lemma 1.64 (Completing the Square). *Assume z is a solution of the Riccati equation (1.5) on $[\rho(a), b]$. Let $u \in \mathbb{A}$. Then for all $t \in [a, b]$, we have*

$$
\begin{aligned}
(zu^2)^\nabla(t) \;=\;& p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t) \\
& - \left[\frac{z^\rho(t)u(t)}{\sqrt{p^\rho(t) + \nu(t)z^\rho(t)}} - \sqrt{p^\rho(t) + \nu(t)z^\rho(t)}\,u^\nabla(t)\right]^2.
\end{aligned}
$$

Proof. Let z be a solution of the Riccati equation (1.5) on $[\rho(a), b]$, and let $u \in \mathbb{A}$. Then for $t \in [a, b]$,

$$
\begin{aligned}
(z(t)u^2(t))^\nabla \;=\;& z^\nabla(t)(u^2(t)) + z^\rho(t)(u^2(t))^\nabla \\
=\;& z(t)^\nabla(u^2(t)) + z^\rho(t)(u^\rho(t)u^\nabla(t) + u(t)u^\nabla(t)) \\
=\;& \left[-q(t) - \frac{(z^\rho(t))^2}{p^\rho(t) + \nu(t)z^\rho(t)}\right]u^2(t) \\
& + z^\rho(t)u^\rho(t)u^\nabla(t) + z^\rho(t)u(t)u^\nabla(t) \\
=\;& -q(t)u^2(t) - \frac{(z^\rho(t))^2u^2(t)}{p^\rho(t) + \nu(t)z^\rho(t)} \\
& + z^\rho(t)u(t)u^\nabla(t) + z^\rho(t)u^\nabla(t)(u(t) - \nu(t)u^\nabla(t)) \\
=\;& -q(t)u^2(t) - \frac{(z^\rho(t))^2u^2(t)}{p^\rho(t) + \nu(t)z^\rho(t)} \\
& + 2z^\rho(t)u(t)u^\nabla(t) - z^\rho(t)\nu(t)(u^\nabla(t))^2 \\
=\;& p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t) - \frac{(z^\rho(t))^2u^2(t)}{p^\rho(t) + \nu(t)z^\rho(t)}
\end{aligned}
$$

$$+2z^\rho(t)u(t)u^\nabla(t) - (p^\rho(t) + z^\rho(t)\nu(t))(u^\nabla(t))^2$$
$$= p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t)$$
$$- \left[\frac{z^\rho(t)u(t)}{\sqrt{p^\rho(t) + \nu(t)z^\rho(t)}} - \sqrt{p^\rho(t) + \nu(t)z^\rho(t)}u^\nabla(t)\right]^2.$$

$\square$

Theorem 1.65. *Let x be a solution of (1.1) on $[\rho(a), \sigma(b)]$, and let $c, d \in [\rho(a), \sigma(b)]$ with $\rho(a) \leq c \leq \sigma(c) < d \leq \sigma(b)$. If $c = \rho(a)$, assume $x(c) = 0$. Now, let*

$$u(t) = \begin{cases} 0 & \rho(a) \leq t < c \\ x(t) & c \leq t < d \\ 0 & d \leq t \leq \sigma(b). \end{cases}$$

Then $u \in \mathbb{A}$, and $\mathcal{F}u = C + D$, where

$$C = \begin{cases} -p(c)x^\Delta(c)x(c) & \nu(c) = 0 \\ \frac{p^\rho(c)x(c)x^\rho(c)}{\nu(c)} & \nu(c) > 0, \end{cases}$$

$$D = \begin{cases} p(d)x^\Delta(d)x(d) & \nu(d) = 0 \\ \frac{p^\rho(d)x(d)x^\rho(d)}{\nu(d)} & \nu(d) > 0. \end{cases}$$

Proof. Let x, u be as described in the statement of the theorem. We first claim that $u \in \mathbb{A}$. It is apparent from the definition that $u \in C^1_{pld}([\rho(a), \sigma(b)], \mathbb{R})$, and that $u(\sigma(b)) = 0$. The fact that $u(\rho(a)) = 0$ is also clear from the definition unless $\rho(a) = c$. In this case, however, $u(\rho(a)) = u(c) = x(c) = 0$, by our assumption on x. So, $u \in \mathbb{A}$, as desired. Now consider

$$\mathcal{F}u = \int_{\rho(a)}^{\sigma(b)} \left[p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t)\right] \nabla t.$$

We have $u(t) = 0$ on $[\rho(a), c) \cup [d, \sigma(b)]$, and $u^\nabla = 0$ on $[\rho(a), c) \cup (d, \sigma(b)]$, so we get

$$\mathcal{F}u = \int_{\rho(c)}^{d} \left[p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t)\right] \nabla t.$$

Breaking up the integral, we get

$$
\begin{aligned}
\mathcal{F}u &= \int_{\rho(c)}^{c} p^{\rho}(t)(u^{\nabla}(t))^2\ \nabla t + \int_{c}^{\rho(d)} p^{\rho}(t)(u^{\nabla}(t))^2\ \nabla t \\
&\quad + \int_{\rho(d)}^{d} p^{\rho}(t)(u^{\nabla}(t))^2\ \nabla t - \int_{\rho(c)}^{c} q(t)u^2(t)\ \nabla t \\
&\quad - \int_{c}^{\rho(d)} q(t)u^2(t)\ \nabla t - \int_{\rho(d)}^{d} q(t)u^2(t)\ \nabla t.
\end{aligned}
$$

Now, we apply Lemma 1.13 to get

$$
\begin{aligned}
\mathcal{F}u &= p^{\rho}(c)(u^{\nabla}(c))^2\nu(c) + p^{\rho}(d)(u^{\nabla}(d))^2\nu(d) \\
&\quad - q(c)u^2(c)\nu(c) - q(d)u^2(d)\nu(d) \\
&\quad + \int_{c}^{\rho(d)} p^{\rho}(t)(u^{\nabla}(t))^2\ \nabla t - \int_{c}^{\rho(d)} q(t)u^2(t)\ \nabla t.
\end{aligned}
$$

Since $u(d) = 0$, the fourth term in this expression vanishes. Furthermore, $u(t) = x(t)$ on $[c, d)$, and $u^{\nabla}(t) = x^{\nabla}(t)$ on (c, d), thus we may substitute x for u in the two remaining integrals. We make this substitution and then evaluate the first of the two remaining integrals by parts, which yields

$$
\begin{aligned}
\mathcal{F}u &= p^{\rho}(c)(u^{\nabla}(c))^2\nu(c) + p^{\rho}(d)(u^{\nabla}(d))^2\nu(d) \\
&\quad - q(c)u^2(c)\nu(c) - q(d)u^2(d)\nu(d) \\
&\quad + \int_{c}^{\rho(d)} p^{\rho}(t)(x^{\nabla}(t))^2\ \nabla t - \int_{c}^{\rho(d)} q(t)x^2(t)\ \nabla t \\
&= p^{\rho}(c)(u^{\nabla}(c))^2\nu(c) + p^{\rho}(d)(u^{\nabla}(d))^2\nu(d) \\
&\quad - q(c)u^2(c)\nu(c) + p(\rho(d))x^{\Delta}(\rho(d))x(\rho(d)) \\
&\quad - p(c)x^{\Delta}(c)x(c) - \int_{c}^{\rho(d)} \left[p(t)x^{\Delta}(t)\right]^{\nabla} x(t)\ \nabla t \\
&\quad + \int_{c}^{\rho(d)} q(t)(x(t))^2\ \nabla t \\
&= p^{\rho}(c)(u^{\nabla}(c))^2\nu(c) + p^{\rho}(d)(u^{\nabla}(d))^2\nu(d) \\
&\quad - q(c)u^2(c)\nu(c) + p(\rho(d))x^{\Delta}(\rho(d))x(\rho(d)) \\
&\quad - p(c)x^{\Delta}(c)x(c) - \int_{c}^{\rho(d)} x(t)Lx(t)\ \nabla t \\
&= p^{\rho}(c)(u^{\nabla}(c))^2\nu(c) + p^{\rho}(d)(u^{\nabla}(d))^2\nu(d) \\
&\quad - q(c)u^2(c)\nu(c) + p(\rho(d))x^{\Delta}(\rho(d))x(\rho(d)) \\
&\quad - p(c)x^{\Delta}(c)x(c) \\
&= C + D,
\end{aligned}
$$

where

$$C = \nu(c)p^\rho(c)(u^\nabla(c))^2 - \nu(c)q(c)(u(c))^2$$
$$-p(c)x^\Delta(c)x(c),$$

and

$$D = \nu(d)p^\rho(d)(u^\nabla(d))^2 + p(\rho(d))x^\Delta(\rho(d))x(\rho(d)).$$

Note that if $\nu(c) = 0$, then $C = -p(c)x^\Delta(c)x(c)$. If $\nu(c) > 0$, then c is left-scattered, so we get

$$
\begin{aligned}
C &= \nu(c)p^\rho(c)\left[\frac{u(c) - u(\rho(c))}{\nu(c)}\right]^2 - \nu(c)q(c)u^2(c) \\
&\quad -p(c)x^\Delta(c)x(c) \\
&= \frac{p^\rho(c)x^2(c)}{\nu(c)} - \nu(c)q(c)x^2(c) - p(c)x^\Delta(c)x(c) \\
&\quad +p^\rho(c)x^{\Delta\rho}(c)x(c) - p^\rho(c)x^{\Delta\rho}(c)x(c) \\
&= \frac{p^\rho(c)x^2(c)}{\nu(c)} - p^\rho(c)x^{\Delta\rho}(c)x(c) \\
&\quad -\nu(c)x(c)\left[q(c)x(c) + \frac{p(c)x^\Delta(c) - p^\rho(c)x^{\Delta\rho}(c)}{\nu(c)}\right] \\
&= \frac{p^\rho(c)x^2(c)}{\nu(c)} - p^\rho(c)x^\nabla(c)x(c) \\
&\quad -\nu(c)x(c)\left(q(c)x(c) + [px^\Delta]^\nabla(c)\right) \\
&= \frac{p^\rho(c)x^2(c)}{\nu(c)} - p^\rho(c)x(c)\left(\frac{x(c) - x^\rho(c)}{\nu(c)}\right) \\
&= \frac{p^\rho(c)x^2(c) - p^\rho(c)x^2(c) + p^\rho(c)x(c)x^\rho(c)}{\nu(c)} \\
&= \frac{p^\rho(c)x(c)x^\rho(c)}{\nu(c)},
\end{aligned}
$$

so C is as described in the statement of the theorem. Now note that if $\nu(d) = 0$, then $D = p(\rho(d))x^\Delta(\rho(d))x(\rho(d)) = p(d)x^\Delta(d)x(d)$. If $\nu(d) > 0$, then d is left-scattered, so we get

$$
\begin{aligned}
D &= \nu(d)p^\rho(d)(u^\nabla(d))^2 + p^\rho(d)x^\nabla(d)x^\rho(d) \\
&= \nu(d)p^\rho(d)\left[\frac{u(d) - u(\rho(d))}{\nu(d)}\right]^2 + p^\rho(d)x^\rho(d)\frac{x(d) - x^\rho(d)}{\nu(d)} \\
&= \frac{p^\rho(d)(x^\rho(d))^2}{\nu(d)} + \frac{p^\rho(d)x^\rho(d)x(d)}{\nu(d)} - \frac{p^\rho(d)(x^\rho(d))^2}{\nu(d)} \\
&= \frac{p^\rho(d)x^\rho(d)x(d)}{\nu(d)}.
\end{aligned}
$$

Thus D is as desired, and the proof is complete. $\square$

Theorem 1.66 (Jacobi's Condition). *The self-adjoint equation* (1.1) *is disconjugate on* $[\rho(a), \sigma(b)]$ *if and only if* $\mathcal{F}$ *is positive definite on* $\mathbb{A}$.

Proof. First, suppose (1.1) is disconjugate on $[\rho(a), \sigma(b)]$. Then there is a positive solution, x, of (1.1) with on $[\rho(a), \sigma(b)]$. Let $z(t) := \frac{p(t)x^{\Delta}(t)}{x(t)}$. Then by Theorem 1.60, z is a solution of $Rz = 0$ on $[\rho(a), b]$. Thus by Lemma 1.64, for any $u \in \mathbb{A}$,

$$
\begin{aligned}
(z(t)u^2(t))^{\nabla} &= p^{\rho}(t)(u^{\nabla}(t))^2 - q(t)u^2(t) \\
&\quad - \left[\frac{z^{\rho}(t)u(t)}{\sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}} - \sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}u^{\nabla}(t) \right]^2
\end{aligned}
$$

for $t \in [a, b]$. In fact, it can be shown that this equation holds at $t = \sigma(b)$ as well. As the equation holds on $[a, \sigma(b)]$, we may integrate from $\rho(a)$ to $\sigma(b)$, and noting that $u(\rho(a)) = u(\sigma(b)) = 0$, we get

$$
\mathcal{F}u = \int_{\rho(a)}^{\sigma(b)} \left[\frac{z^{\rho}(t)u(t)}{\sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}} - \sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}u^{\nabla}(t) \right]^2 \nabla t,
$$

so $\mathcal{F}u \geq 0$ for all $u \in \mathbb{A}$. Furthermore, it is clear that if $u \equiv 0$, then $\mathcal{F}u = 0$. Now suppose $\mathcal{F}u = 0$. Then

$$
\frac{z^{\rho}(t)u(t)}{\sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}} = \sqrt{p^{\rho}(t) + \nu(t)z^{\rho}(t)}u^{\nabla}(t),
$$

so u solves the initial value problem

$$
u^{\nabla} = \frac{z^{\rho}}{p^{\rho} + \nu z^{\rho}} u, \quad u(\sigma(b)) = 0
$$

on $[a, \sigma(b)]$ Since $\frac{z^{\rho}}{p^{\rho} + \nu z^{\rho}} \in \mathcal{R}_{\nu}^{+}$, the solution of this IVP is unique, and gives $u(t) \equiv 0$ on $[a, \sigma(b)]$. As $u(\rho(a)) = 0$ as well, we get $u(t) \equiv 0$ on $[\rho(a), \sigma(b)]$. Hence, $\mathcal{F}$ is positive definite on $\mathbb{A}$.

We will prove the converse of this statement by contrapositive. Suppose (1.1) is not disconjugate on $[\rho(a), \sigma(b)]$. Then there is a nontrivial solution x of (1.1) such that either $x(\rho(a)) = 0$ and x has a generalized zero in $(\rho(a), \sigma(b)]$, or x has two generalized zeros in $(\rho(a), \sigma(b)]$. In either case, let $c < d$ be the two smallest generalized zeros of x in $[\rho(a), \sigma(b)]$. Then, let

$$
u(t) = \begin{cases} 0 & \rho(a) \leq t < c \\[2mm] x(t) & c \leq t < d \\[2mm] 0 & d \leq t \leq \sigma(b). \end{cases}
$$

Applying Theorem 1.65, we then have $\mathcal{F} = C + D \leq 0$. As u is not identically 0, this tells us that $\mathcal{F}$ is not positive definite. By contrapositive, the proof is complete. $\qquad \square$

Theorem 1.67 (Sturm Comparison Theorem). *Let*

$$L_1 x = [p_1(t)x^\Delta]^\nabla + q_1(t)x,$$

$$L_2 x = [p_2(t)x^\Delta]^\nabla + q_2(t)x.$$

Assume $q_1(t) \geq q_2(t)$ and $0 < p_1(t) \leq p_2(t)$ for $t \in [\rho(a), \sigma(b)]$. If $L_1 x(t) = 0$ is disconjugate on $[\rho(a), \sigma(b)]$, then $L_2 x(t) = 0$ is disconjugate on $[\rho(a), \sigma(b)]$.

Proof. Let

$$\mathcal{F}_1(u) := \int_{\rho(a)}^{\sigma(b)} \left[p_1^\rho(t)(u^\nabla(t))^2 - q_1(t)u^2(t) \right] \nabla t,$$

$$\mathcal{F}_2(u) := \int_{\rho(a)}^{\sigma(b)} \left[p_2^\rho(t)(u^\nabla(t))^2 - q_2(t)u^2(t) \right] \nabla t.$$

Assume that $L_1 x(t) = 0$ is disconjugate on $[\rho(a), \sigma(b)]$. Then by Theorem 1.66, the quadratic functional $\mathcal{F}_1$ is positive definite on $\mathbb{A}$. Then, for $u \in \mathbb{A}$, we have

$$
\begin{aligned}
\mathcal{F}_2 u &= \int_{\rho(a)}^{\sigma(b)} \left[p_2^\rho(t)(u^\nabla(t))^2 - q_2(t)u^2(t) \right] \nabla t \\
&\geq \int_{\rho(a)}^{\sigma(b)} \left[p_1^\rho(t)(u^\nabla(t))^2 - q_1(t)u^2(t) \right] \nabla t \\
&= \mathcal{F}_1 u.
\end{aligned}
$$

Hence $\mathcal{F}_2$ is positive definite on $\mathbb{A}$. Then, again by Theorem 1.66, $L_2 x = 0$ is disconjugate on $[\rho(a), \sigma(b)]$. $\qquad \square$

Theorem 1.68. *Let $L_1 x$, $L_2 x$ be as in the Sturm Comparison Theorem. Then if $L_i x = 0$ is disconjugate on $[\rho(a), \sigma(b)]$ for $i = 1, 2$, and if*

$$p(t) = \lambda_1 p_1(t) + \lambda_2 p_2(t), \quad and$$

$$q(t) = \lambda_1 q_1(t) + \lambda_2 q_2(t),$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, then $Lx = 0$ is disconjugate on $[\rho(a), \sigma(b)]$.

Proof. Suppose $L_i x = 0$ is disconjugate on $[\rho(a), \sigma(b)]$ for $i = 1, 2$. Then the quadratic functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ are positive definite on $\mathbb{A}$. Then for $u \in \mathbb{A}$, we

have

$$
\begin{aligned}
\mathcal{F}u &= \int_{\rho(a)}^{\sigma(b)} \left[p^\rho(t)(u^\nabla(t))^2 - q(t)u^2(t) \right] \nabla t \\
&= \int_{\rho(a)}^{\sigma(b)} \left[(\lambda_1 p_1(t) + \lambda_2 p_2(t))(u^\nabla(t))^2 - (\lambda_1 q_1(t) + \lambda_2 q_2(t))u^2(t) \right] \nabla t \\
&= \int_{\rho(a)}^{\sigma(b)} \left[p_1^\rho(t)(u^\nabla(t))^2 - q_1(t)u^2(t) \right] \nabla t \\
&\quad + \int_{\rho(a)}^{\sigma(b)} \left[p_2^\rho(t)(u^\nabla(t))^2 - q_2(t)u^2(t) \right] \nabla t \\
&= \mathcal{F}_1 u + \mathcal{F}_2 u.
\end{aligned}
$$

Therefore, $\mathcal{F}$ is positive definite on $\mathbb{A}$, and hence $Lx = 0$ is disconjugate on $[\rho(a), \sigma(b)]$.   □

We summarize some of the major results in the following theorem.

Theorem 1.69 (Reid Roundabout Theorem). *The following are equivalent:*

(i) $Lx = 0$ is disconjugate on $[\rho(a), \sigma(b)]$;

(ii) $Lx = 0$ has a positive solution on $[\rho(a), \sigma(b)]$;

(iii) The quadratic functional $\mathcal{F}$ is positive definite on $\mathbb{A}$;

(iv) the Riccati differential inequality $Rz \leq 0$ has a solution on $[\rho(a), b]$.

Proof. By Theorem 1.49, (i) and (ii) are equivalent. By Theorem 1.66, (i) and (iii) are equivalent. By Theorem 1.60, (ii) implies (iv). It remains to show that (iv) implies (i). So, assume $Rz \leq 0$ has a solution, z on $[\rho(a), b]$, and let

$$
w(t) := Rz(t) \quad \text{for} \quad t \in [a, b].
$$

If $\rho(a) < a$, let $w(\rho(a)) = 0$, and if $\sigma(b) > b$, let $w(\sigma(b)) = 0$. Then z is a solution of the Riccati equation

$$
z^\nabla(t) + (q(t) - w(t)) + \frac{(z^\rho(t))^2}{p^\rho(t) + \nu(t)z^\rho(t)} = 0
$$

on $[\rho(a), b]$. This implies that the self-adjoint dynamic equation

$$
(p(t)x^\Delta)^\nabla + (q(t) - w(t))x = 0
$$

is disconjugate on $[\rho(a), \sigma(b)]$. But

$$
q(t) - w(t) \geq q(t)
$$

on $[\rho(a), \sigma(b)]$, so by Theorem 1.67, $Lx = 0$ is disconjugate on $[\rho(a), \sigma(b)]$, and the proof is complete.   □

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A SECOND-ORDER SELF-ADJOINT EQUATION ON A TIME SCALE

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ABSTRACT. In this paper, we examine the dynamic equation $[p(t)x^\Delta(t)]^\nabla + q(t)x(t) = 0$ on a time scale. Little work has been done on this equation, which combines both the delta and nabla derivatives. Several preliminary results are established, including Abel's Formula and its Converse. We then proceed to investigate oscillation and disconjugacy of this dynamic equation.

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1. NABLA DERIVATIVES

In this paper, we are concerned with the second-order self-adjoint dynamic equation

$$[p(t)x^\Delta]^\nabla + q(t)x = 0.$$

We begin our work by reviewing some properties of the nabla derivative, and then in section 2, we proceed to establish several results concerning the interaction of the two types of derivatives. In the third section of the paper, we develop Abel's Formula, and its converse, which we then use to prove a reduction of order theorem. In the final section, we turn our attention to oscillation and disconjugacy, establishing first an analogue of the Sturm Separation Theorem, and then, via the Polya and Trench factorizations, we demonstrate the existence of recessive and dominant solutions of the self-adjoint equation.

Here, it is assumed that the reader is already familiar with the basic notions of calculus on a time scale, using the delta-derivative (or Δ-derivative). The reader may be less familiar, however, with the nabla-derivative (or ∇-derivative) on a time scale, as developed by Atici and Guseinov [1], and so we include here a brief introduction to its properties, as previously established in other works, stating them without proof. Readers desiring more information are directed to [2] and [1].

Throughout, we assume that $\mathbb{T}$ is a time scale. The notation $[a, b]$ is understood to mean the real interval $[a, b]$ intersected with $\mathbb{T}$.

Definition 1.1. Let $\mathbb{T}$ be a time scale. For $t > \inf(\mathbb{T})$, the *backward jump operator*, $\rho(t)$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

and the *backward graininess function* $\nu(t)$ is defined by

$$\nu(t) := t - \rho(t).$$

If $f : \mathbb{T} \to \mathbb{R}$, the notation $f^\rho(t)$ is understood to mean $f(\rho(t))$.

Remark 1.2. Here, we retain the original definition of $\nu(t)$. This definition is consistent with the original literature published on ∇-derivatives. It is inconsistent, however with the current work on α-derivatives. When working with α-derivatives, the α-graininess, μ_α is defined to be $\mu_\alpha := \alpha(t) - t$. When $\alpha(t) = \rho(t)$, then, we would have $\mu_\rho := \rho(t) - t = -\nu(t)$. This inconsistency is unfortunate, but we feel it is more important that we remain consistent with the way $\nu(t)$ was defined in previously published work. To minimize confusion, we recommend the notation $\mu_\rho(t) = \rho(t) - t$ be used in work that is to be interpreted in the more general α-derivative setting.

Definition 1.3. Define the set $\mathbb{T}_\kappa$ as follows: If $\mathbb{T}$ has a right-scattered minimum m, set $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$.

Definition 1.4. Let $t \in \mathbb{T}_\kappa$. Then the ∇-*derivative of f at t*, denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \le \varepsilon|\rho(t) - s|$$

for all $s \in U$.

For $\mathbb{T} = \mathbb{R}$, the ∇-derivative is just the usual derivative. That is, $f^\nabla = f'$. For $\mathbb{T} = \mathbb{Z}$ the ∇-derivative is the backward difference operator, $f^\nabla(t) = \nabla f(t) := f(t) - f(t-1)$.

Definition 1.5. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left-dense continuous* or *ld-continous* if it is continuous at left-dense points, and if its right-sided limit exists (finite) at right-dense points.

Definition 1.6. It can be shown that if f is ld-continuous then there is a function F, called a ∇-*antiderivative*, such that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}$. We then define the ∇-*integral* (∇-Cauchy integral) of f by

$$\int_{t_0}^t f(s)\, \nabla s = F(t) - F(t_0).$$

Theorem 1.7. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$. Then we have the following:*

1. *If f is nabla-differentiable at t, then f is continuous at t.*

2. *If f is continuous at t and t is left-scattered, then f is nabla-differentiable at t with*

$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

3. *If t is left-dense, then f is nabla-differentiable at t iff the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

4. *If f is nabla-differentiable at t, then*

$$f^{\rho}(t) = f(t) - \nu(t)f^{\nabla}(t).$$

Other properties of both the ∇-derivative and the ∇-integral are analogous to the properties of the Δ-derivative and Δ-integral. For example, both differentiation and integration are linear operations, and there are product and quotient rules for differentiation, as well as integration by parts formulas. Readers interested in the specifics can find more details in [2].

2. PRELIMINARY RESULTS

We are interested in the second-order self-adjoint dynamic equation

(2.1) $\qquad\qquad Lx = 0 \quad \text{where} \quad Lx = [p(t)x^{\Delta}]^{\nabla} + q(t)x.$

Here we assume that p is continuous, q is ld-continuous and that

$$p(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

Define the set $\mathbb{D}$ to be the set of all functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous and such that $[p(t)x^{\Delta}]^{\nabla} : \mathbb{T}^{\kappa}_{\kappa} \to \mathbb{R}$ is ld-continuous. A function $x \in \mathbb{D}$ is said to be a solution of $Lx = 0$ on $\mathbb{T}$ provided $Lx(t) = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Since the equation we are interested in, equation (2.1), contains both Δ- and ∇-derivatives, we establish here some results regarding the relationship between these two types of derivatives on time scales.

One of the following results relies on L'Hôpital's rule. A version of L'Hôpital's rule involving Δ-derivatives is contained in [2]. We state its analog for ∇-derivatives here. As we may wish to use L'Hôpital's rule to evaluate a limit as $t \to \pm\infty$, we make the following definition.

Definition 2.1. Let $\varepsilon > 0$. If $\mathbb{T}$ is unbounded below, we define a *right neighborhood of* $-\infty$, denoted $R_\varepsilon(-\infty)$ by

$$R_\varepsilon(-\infty) = \left\{ t \in \mathbb{T} : t < -\frac{1}{\varepsilon} \right\}.$$

We next define a right neighborhood for points in $\mathbb{T}$.

Definition 2.2. Let $\varepsilon > 0$. For any right-dense $t_0 \in \mathbb{T}$, define a *right neighborhood of* t_0, denoted $R_\varepsilon(t_0)$, by

$$R_\varepsilon(t_0) := \{ t \in \mathbb{T} : 0 < t - t_0 < \varepsilon \}.$$

Theorem 2.3 (L'Hôpital's Rule). *Assume f and g are ∇-differentiable on $\mathbb{T}$ and let $t_0 \in \mathbb{T} \cup \{-\infty\}$. If $t_0 \in \mathbb{T}$, assume t_0 is right-dense. Furthermore, assume*

$$\lim_{t \to t_0^+} f(t) = \lim_{t \to t_0^+} g(t) = 0,$$

and suppose there exists $\varepsilon > 0$ such that $g(t)g^\nabla(t) > 0$ for all $t \in R_\varepsilon(t_0)$. Then

$$\liminf_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)} \le \liminf_{t \to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^+} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^+} \frac{f^\nabla(t)}{g^\nabla(t)}.$$

Proof. Without loss of generality, assume $g(t)$ and $g^\nabla(t)$ are both strictly positive on $R_\varepsilon(t_0)$.

Let $\delta \in (0, \varepsilon]$, and let $a := \inf_{\tau \in R_\delta(t_0)} \frac{f^\nabla(\tau)}{g^\nabla(\tau)}$, $b := \sup_{\tau \in R_\delta(t_0)} \frac{f^\nabla(\tau)}{g^\nabla(\tau)}$. To complete the proof, it suffices to show

$$a \le \inf_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \le \sup_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \le b,$$

as we may then let $\delta \to 0$ to obtain the desired result.

We must be careful here, as either a or b could possibly be infinite. Note, however, that since $g^\nabla(\tau) > 0$ on $R_\delta(t_0)$, we have $a < \infty$. Similarly, $b > -\infty$. So our only concern is if $a = -\infty$ or $b = \infty$. But, if $a = -\infty$, we have immediately that

$$a \le \inf_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)},$$

as desired, and if $b = \infty$ we have immediately that

$$\sup_{\tau \in R_\delta(t_0)} \frac{f(\tau)}{g(\tau)} \le b,$$

as desired. Therefore, we may assume that both a and b are finite. Then

$$ag^\nabla(\tau) \le f^\nabla(\tau) \le bg^\nabla(\tau) \quad \text{for all} \quad \tau \in R_\delta(t_0),$$

and by a theorem of Guseinov and Kaymakçalan [3],

$$\int_t^s ag^\nabla(\tau)\, \nabla\tau \le \int_t^s f^\nabla(\tau)\, \nabla\tau \le \int_t^s bg^\nabla(\tau)\, \nabla\tau \quad \text{for all} \quad s, t \in R_\delta(t_0),\ t < s.$$

Integrating, we see that

$$ag(s) - ag(t) \leq f(s) - f(t) \leq bg(s) - bg(t) \quad \text{for all} \quad s, t \in R_\delta(t_0), \ t < s.$$

Letting $t \to t_0^+$, we get

$$ag(s) \leq f(s) \leq bg(s) \quad \text{for all} \quad s \in R_\delta(t_0),$$

and thus

$$a \leq \inf_{s \in R_\delta(t_0)} \frac{f(s)}{g(s)} \leq \sup_{s \in R_\delta(t_0)} \frac{f(s)}{g(s)} \leq b.$$

Then, by the discussion above, the proof is complete. $\square$

Remark 2.4. Although the preceding theorem is only stated in terms of one-sided limits, an analogous result can be established if the limit is taken from the other direction. Left neighborhoods of ∞ or of points in $\mathbb{T}$ are defined in similar manner to right neighborhoods. To apply L'Hôpital's rule using a left-sided limit, t_0 must be left-dense (or ∞ if $\mathbb{T}$ is unbounded above), and gg^∇ must be strictly negative on some left neighborhood of t_0.

In order to determine when the two types of derivatives may be interchanged, we need to consider some of the points in our time scale separately, so let

$$A := \{t \in \mathbb{T} \mid t \text{ is left-dense and right-scattered}\}, \quad \mathbb{T}_A := \mathbb{T} \setminus A.$$

Additionally, let

$$B := \{t \in \mathbb{T} \mid t \text{ is right-dense and left-scattered}\}, \quad \mathbb{T}_B := \mathbb{T} \setminus B.$$

The following lemma is very easy to prove, and we omit the proof here.

Lemma 2.5. *If $t \in \mathbb{T}_A$ then $\sigma(\rho(t)) = t$. If $t \in \mathbb{T}_B$ then $\rho(\sigma(t)) = t$.*

Theorem 2.6. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and f^Δ is rd-continuous on $\mathbb{T}^\kappa$ then f is ∇-differentiable on $\mathbb{T}_\kappa$, and*

$$f^\nabla(t) = \begin{cases} f^\Delta(\rho(t)) & t \in \mathbb{T}_A \\ \lim_{s \to t^-} f^\Delta(s) & t \in A. \end{cases}$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_\kappa$ and g^∇ is ld-continuous on $\mathbb{T}_\kappa$ then g is Δ-differentiable on $\mathbb{T}^\kappa$, and

$$g^\Delta(t) = \begin{cases} g^\nabla(\sigma(t)) & t \in \mathbb{T}_B \\ \lim_{s \to t^+} g^\nabla(s) & t \in B. \end{cases}$$

Proof. We will only prove the first statement. The proof of the second statement is similar. First, assume $t \in \mathbb{T}_A$. Then there are two cases: Either

1. t is left-scattered, or
2. t is both left-dense and right-dense.

Case 1: Suppose t is left-scattered and f is Δ-differentiable on $\mathbb{T}^\kappa$. Then f is continuous at t, and is therefore ∇-differentiable at t. Next, note that $\rho(t)$ is right-scattered, and

$$
\begin{aligned}
f^\Delta(\rho(t)) &= \frac{f(\sigma(\rho(t))) - f(\rho(t))}{\sigma(\rho(t)) - \rho(t)} \\
&= \frac{f(t) - f(\rho(t))}{t - \rho(t)} \\
&= f^\nabla(t).
\end{aligned}
$$

Case 2: Now, suppose t is both left-dense and right-dense, and $f : \mathbb{T} \to \mathbb{R}$ is continuous on $\mathbb{T}$ and Δ-differentiable at t. Since t is right-dense and f is Δ-differentiable at t, we have that

$$
\lim_{s \to t} \frac{f(t) - f(s)}{t - s}
$$

exists. But t is left-dense as well, so this expression also defines $f^\nabla(t)$, and we see that

$$
\begin{aligned}
f^\nabla(t) &= \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \\
&= f^\Delta(t) \\
&= f^\Delta(\rho(t)).
\end{aligned}
$$

So, we have established the desired result in the case where $t \in \mathbb{T}_A$.

Now suppose $t \in A$. Then t is left-dense. Hence $f^\nabla(t)$ exists provided

$$
\lim_{s \to t} \frac{f(t) - f(s)}{t - s}
$$

exists.

As t is right-scattered, we need only consider the limit as $s \to t$ from the left. Then we apply L'Hopital's rule [2], differentiating with respect to s to get

$$
\lim_{s \to t-} \frac{f(t) - f(s)}{t - s} = \lim_{s \to t-} \frac{-f^\Delta(s)}{-1} = \lim_{s \to t-} f^\Delta(s).
$$

Since we have assumed that f^Δ is rd-continuous, this limit exists. Hence f is ∇-differentiable, and $f^\nabla(t) = \lim_{s \to t-} f^\Delta(t)$, as desired. $\qquad\square$

Corollary 2.7. *If $t_0 \in \mathbb{T}$, and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$ then $\int_{t_0}^t f(\tau)\Delta\tau$ is ∇-differentiable on $\mathbb{T}$ and*

$$
\left[\int_{t_0}^t f(\tau)\Delta\tau \right]^\nabla = \begin{cases} f(\rho(t)) & \text{if } t \in \mathbb{T}_A \\ \lim_{s \to t-} f(s) & t \in A. \end{cases}
$$

If $t_0 \in \mathbb{T}$, and $g : \mathbb{T} \to \mathbb{R}$ is ld-continuous on $\mathbb{T}$ then $\int_{t_0}^t g(\tau)\nabla\tau$ is Δ-differentiable on $\mathbb{T}$ and

$$
\left[\int_{t_0}^t g(\tau)\nabla\tau \right]^\Delta = \begin{cases} g(\sigma(t)) & \text{if } t \in \mathbb{T}_B \\ \lim_{s \to t+} g(s) & t \in B. \end{cases}
$$

The following corollary was previously established by Atici and Guseinov in their work [1].

Corollary 2.8. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and if f^Δ is continuous on $\mathbb{T}^\kappa$, then f is ∇-differentiable on $\mathbb{T}_\kappa$ and*

$$f^\nabla(t) = f^{\Delta\rho}(t) \quad \text{for } t \in \mathbb{T}_\kappa.$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}^\kappa$ and if g^∇ is continuous on $\mathbb{T}_\kappa$, then g is Δ-differentiable on $\mathbb{T}^\kappa$ and

$$g^\Delta(t) = g^{\nabla\sigma}(t) \quad \text{for } t \in \mathbb{T}^\kappa.$$

3. ABEL'S FORMULA AND REDUCTION OF ORDER

We begin this section by looking at the Lagrange Identity for the dynamic equation (2.1). We establish several corollaries and related results, including Abel's Formula and its converse. We conclude the section with a reduction of order theorem. Some of the results in this section are due to Atici and Guseinov. Specifically, Theorem 3.1 and Corollary 3.5 were previously established in their work [1]. Our conditions on p and q are less restrictive than Atici and Guseinov's, and our domain of interest, $\mathbb{D}$, is defined more broadly. In spite of this, however, many of the proofs contained in [1] remain valid. As this is the case, we have omitted the proofs of some of the following theorems, and refer the reader to Atici and Guseinov's work.

Theorem 3.1. *If $t_0 \in \mathbb{T}$, and x_0 and x_1 are given constants, then the initial value problem*

$$Lx = 0, \quad x(t_0) = x_0, \quad x^\Delta(t_0) = x_1$$

has a unique solution, and this solution exists on all of $\mathbb{T}$.

Definition 3.2. *If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the Wronskian of x and y, denoted $W(x, y)(t)$ is defined by*

$$W(x, y)(t) = \begin{vmatrix} x(t) & y(t) \\ x^\Delta(t) & y^\Delta(t) \end{vmatrix} \quad \text{for } t \in \mathbb{T}^\kappa.$$

Definition 3.3. *If x, y are Δ-differentiable on $\mathbb{T}^\kappa$, then the Lagrange bracket of x and y is defined by*

$$\{x; y\}(t) = p(t)W(x, y)(t) \quad \text{for } t \in \mathbb{T}^\kappa.$$

Theorem 3.4 (Lagrange Identity). *If $x, y \in \mathbb{D}$, then*

$$x(t)Ly(t) - y(t)Lx(t) = \{x; y\}^\nabla(t) \quad \text{for } t \in \mathbb{T}_\kappa^\kappa.$$

Proof. Let $x, y \in \mathbb{D}$. We have

$$
\begin{aligned}
\{x; y\}^\nabla &= [pW(x,y)]^\nabla \\
&= [xpy^\Delta - ypx^\Delta]^\nabla \\
&= x^\nabla p^\rho y^{\Delta\rho} + x[py^\Delta]^\nabla - y^\nabla p^\rho x^{\Delta\rho} - y[px^\Delta]^\nabla \\
&= x^\nabla p^\rho y^\nabla + x[py^\Delta]^\nabla - y^\nabla p^\rho x^\nabla - y[px^\Delta]^\nabla \\
&= x[py^\Delta]^\nabla - y[px^\Delta]^\nabla \\
&= x([py^\Delta]^\nabla + qy) - y([px^\Delta]^\nabla + qx) \\
&= xLy - yLx,
\end{aligned}
$$

where we have made use of the fact that x^Δ and y^Δ are continuous and applied Corollary 2.8. $\qquad\square$

Corollary 3.5 (Abel's Formula). *If x, y are solutions of (2.1) then*

$$W(x,y)(t) = \frac{C}{p(t)} \quad \text{for } t \in \mathbb{T}^\kappa,$$

where C is a constant.

Definition 3.6. Define the *inner product* of x and y on $[a, b]$ by

$$\langle x, y \rangle := \int_a^b x(t)y(t)\nabla t.$$

Corollary 3.7 (Green's Formula). *If $x, y \in \mathbb{D}$ then*

$$\langle x, Ly \rangle - \langle Lx, y \rangle = [p(t)W(x,y)]_a^b.$$

Theorem 3.8 (Converse of Abel's Formula). *Assume u is a solution of (2.1) with $u(t) \neq 0$ for $t \in \mathbb{T}$. If $v \in \mathbb{D}$ satisfies*

$$W(u,v)(t) = \frac{C}{p(t)},$$

then v is also a solution of (2.1).

Proof. Suppose that u is a solution of (2.1) with $u(t) \neq 0$ for any t, and assume that $v \in \mathbb{D}$ satisfies $W(u,v)(t) = \frac{C}{p(t)}$. Then by Theorem 3.4, we have

$$
\begin{aligned}
u(t)Lv(t) - v(t)Lu(t) &= \{u; v\}^\nabla(t) \\
u(t)Lv(t) &= [p(t)W(u,v)(t)]^\nabla \\
&= [p(t)\frac{C}{p(t)}]^\nabla \\
&= C^\nabla \\
&= 0.
\end{aligned}
$$

As $u(t) \neq 0$ for any t, we can divide through by it to get

$$Lv(t) = 0 \quad \text{for } t \in \mathbb{T}_\kappa^\kappa.$$

Hence v is a solution of (2.1) on $\mathbb{T}$.   □

Theorem 3.9 (Reduction of Order). *Let $t_0 \in \mathbb{T}^\kappa$, and assume u is a solution of* (2.1) *with $u(t) \neq 0$ for any t. Then a second, linearly independent solution, v, of* (2.1) *is given by*

$$v(t) = u(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s$$

for $t \in \mathbb{T}$.

Proof. By Theorem 3.8, we need only show that $v \in \mathbb{D}$ and that $W(u,v)(t) = \frac{C}{p(t)}$ for some constant C. Consider first

$$
\begin{aligned}
W(u,v)(t) &= u(t)v^\Delta(t) - v(t)u^\Delta(t) \\
&= u(t)\left[u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s + \frac{u^\sigma(t)}{p(t)u(t)u^\sigma(t)} \right] \\
&\quad - u^\Delta(t)u(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s \\
&= u(t)u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s + \frac{u(t)u^\sigma(t)}{p(t)u(t)u^\sigma(t)} \\
&\quad - u(t)u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s \\
&= \frac{1}{p(t)}.
\end{aligned}
$$

Here we have $C = 1$. It remains to show that $v \in \mathbb{D}$. We have that

$$
\begin{aligned}
v^\Delta(t) &= u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s + \frac{u^\sigma(t)}{p(t)u(t)u^\sigma(t)} \\
&= u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s + \frac{1}{p(t)u(t)}.
\end{aligned}
$$

Since $u \in \mathbb{D}, u(t) \neq 0$ and p is continuous, we have that v^Δ is continuous. Next, consider

$$
\begin{aligned}
[p(t)v^\Delta(t)]^\nabla &= \left[p(t)u^\Delta(t) \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s \right]^\nabla + \left[\frac{1}{u(t)} \right]^\nabla \\
&= [p(t)u^\Delta(t)]^\nabla \int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s \\
&\quad + p^\rho(t)u^{\Delta\rho}(t)\left[\int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s \right]^\nabla - \frac{u^\Delta(t)}{u(t)u^\rho(t)}.
\end{aligned}
$$

Now, the first and last terms are ld-continuous. It is not as clear that the center term is ld-continuous. Specifically, we are concerned about whether or not the expression

$$\left[\int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)} \Delta s \right]^\nabla$$

is ld-continuous. Note that the integrand is rd-continuous. Hence Corollary 2.7 applies and yields

$$\left[\int_{t_0}^t \frac{1}{p(\tau)u(\tau)u^\sigma(\tau)} \Delta\tau \right]^\nabla = \begin{cases} \frac{1}{p^\rho(t)u^\rho(t)u^{\sigma\rho}(t)} & \text{if } t \in \mathbb{T}_A \\ \lim_{s\to t-} \frac{1}{p(s)u(s)u^\sigma(s)} & t \in A. \end{cases}$$

Simplification of this expression gives

$$\left[\int_{t_0}^t \frac{1}{p(\tau)u(\tau)u^\sigma(\tau)} \Delta\tau \right]^\nabla = \frac{1}{p^\rho(t)u^\rho(t)u(t)} \quad \text{for } t \in \mathbb{T}.$$

This function is ld-continuous, and so we have that $v \in \mathbb{D}$. Hence by Theorem 3.8, v is also a solution of (2.1). Finally, note that as $W(u,v)(t) = \frac{1}{p(t)} \neq 0$ for any t, u and v are linearly independent. $\qquad\square$

4. OSCILLATION AND DISCONJUGACY

In this section, we establish results concerning generalized zeros of solutions of (2.1), and examine disconjugacy and oscillation of solutions.

Definition 4.1. We say that a solution, x, of (2.1) has a *generalized zero* at t if

$$x(t) = 0$$

or, if t is left-scattered and

$$x(\rho(t))x(t) < 0.$$

Definition 4.2. We say that (2.1) is *disconjugate* on an interval $[a,b]$ if the following hold.

1. If x is a nontrivial solution of (2.1) with $x(a) = 0$, then x has no generalized zeros in $(a,b]$.
2. If x is a nontrivial solution of (2.1) with $x(a) \neq 0$, then x has at most one generalized zero in $(a,b]$.

We will investigate oscillation of (2.1) as t approaches the supremum of the time scale. Let $\omega = \sup\mathbb{T}$. If $\omega < \infty$, we assume $\rho(\omega) = \omega$. Furthermore, if $\omega < \infty$, we allow the possibility that ω is a singular point for p or q.

Definition 4.3. Let $\omega = \sup\mathbb{T}$ be as described above, and let $a \in \mathbb{T}$. We say that (2.1) is *oscillatory* on $[a,\omega)$ if every nontrivial real-valued solution has infinitely many generalized zeros in $[a,\omega)$. We say (2.1) is *nonoscillatory* if it is not oscillatory.

The following Lemma is a direct consequence of the definition of nonoscillatory.

Lemma 4.4. *Let $\omega = \sup \mathbb{T}$ be as described above, and let $a \in \mathbb{T}$. Then if (2.1) is nonoscillatory on $[a, \omega)$, there is some $t_0 \in \mathbb{T}$, $t_0 \geq a$, such that (2.1) has a positive solution on $[t_0, \omega)$.*

Theorem 4.5 (Sturm Separation Theorem). *Let u and v be linearly independent solution of (2.1). Then u and v have no common zeros in $\mathbb{T}^\kappa$. If u has a zero at $t_1 \in \mathbb{T}$, and a generalized zero at $t_2 > t_1 \in \mathbb{T}$, then v has a generalized zero in $(t_1, t_2]$. If u has generalized zeros at $t_1 \in \mathbb{T}$ and $t_2 > t_1 \in \mathbb{T}$, then v has a generalized zero in $[t_1, t_2]$.*

Proof. If u and v have a common zero at $t_0 \in \mathbb{T}^\kappa$, then

$$W(u, v)(t_0) = \begin{vmatrix} u(t_0) & v(t_0) \\ u^\Delta(t_0) & v^\Delta(t_0) \end{vmatrix} = 0.$$

Hence u and v are linearly dependent.

Now suppose u has a zero at $t_1 \in \mathbb{T}$, and a generalized zero at $t_2 > t_1 \in \mathbb{T}$. Without loss of generality, we may assume $t_2 > \sigma(t_1)$ is the first generalized zero to the right of t_1, $u(t) > 0$ on (t_1, t_2), and $u(t_2) \leq 0$. Assume v is a linearly independent solution of (2.1) with no generalized zero in $(t_1, t_2]$. Without loss of generality, $v(t) > 0$ on $[t_1, t_2]$.

Then on $[t_1, t_2]$,

$$\left(\frac{u}{v}\right)^\Delta (t) = \frac{v(t)u^\Delta(t) - u(t)v^\Delta(t)}{v(t)v^\sigma(t)} = \frac{C}{p(t)v(t)v^\sigma(t)},$$

which is of one sign on $[t_1, t_2]$. Thus $\frac{u}{v}$ is monotone on $[t_1, t_2]$. Fix $t_3 \in (t_1, t_2)$. Note that

$$\frac{u(t_1)}{v(t_1)} = 0, \text{ and } \frac{u(t_3)}{v(t_3)} > 0.$$

But

$$\frac{u(t_2)}{v(t_2)} \leq 0,$$

which contradicts the fact that $\frac{u}{v}$ is monotone on $[t_1, t_2]$. Hence v must have a generalized zero in $(t_1, t_2]$.

Finally, suppose u has generalized zeros at $t_1 \in \mathbb{T}$ and $t_2 > t_1 \in \mathbb{T}$. Assume $t_2 > \sigma(t_1)$ is the first generalized zero to the right of t_1. If $u(t_1) = 0$, we are in the previous case, so assume $u(t_1) \neq 0$. Then, as u has a generalized zero at t_1, we have that t_1 is left-scattered. Without loss of generality, we may assume $u(t) > 0$ on $[t_1, t_2)$, $u(\rho(t_1)) < 0$ and $u(t_2) \leq 0$. Assume v is a linearly independent solution of (2.1) with no generalized zero in $[t_1, t_2)$. Without loss of generality, $v(t) > 0$ on

$[t_1, t_2]$, and $v(\rho(t_1)) > 0$. In a similar fashion to the previous case, we apply Abel's Formula to get that $\frac{u}{v}$ is monotone on $[\rho(t_1), t_2]$. But

$$\frac{u(\rho(t_1))}{v(\rho(t_1))} < 0, \quad \frac{u(t_1)}{v(t_1)} > 0, \quad \text{and} \quad \frac{u(t_2)}{v(t_2)} \leq 0,$$

which is a contradiction. Hence v must have a generalized zero in $[t_1, t_2]$. □

Theorem 4.6. *If* (2.1) *has a positive solution on an interval* $\mathcal{I} \subset \mathbb{T}$ *then* (2.1) *is disconjugate on* $\mathcal{I}$. *Conversely, if* $a, b \in \mathbb{T}_\kappa^\kappa$ *and* (2.1) *is disconjugate on* $[\rho(a), \sigma(b)] \subset \mathbb{T}$, *then* (2.1) *has a positive solution on* $[\rho(a), \sigma(b)]$.

Proof. If (2.1) has a positive solution, u on $\mathcal{I} \subset \mathbb{T}$, then disconjugacy follows from the Sturm Separation Theorem.

Conversely, if (2.1) is disconjugate on the compact interval $[\rho(a), \sigma(b)]$, then let u, v be the solutions of (2.1) satisfying $u(\rho(a)) = 0, u^\Delta(\rho(a)) = 1$ and $v(\sigma(b)) = 0, v^\Delta(b) = -1$. Since (2.1) is disconjugate on $[\rho(a), \sigma(b)]$, we have that $u(t) > 0$ on $(\rho(a), \sigma(b)]$, and $v(t) > 0$ on $[\rho(a), \sigma(b))$. Then

$$x(t) = u(t) + v(t)$$

is the desired positive solution. □

Theorem 4.7 (Polya Factorization). *If* (2.1) *has a positive solution,* u, *on an interval* $\mathcal{I} \subset \mathbb{T}$, *then for any* $x \in \mathbb{D}$, *we get the* Polya Factorization

$$Lx = \alpha_1(t)\{\alpha_2[\alpha_1 x]^\Delta\}^\nabla(t) \text{ for } t \in \mathcal{I},$$

where

$$\alpha_1 := \frac{1}{u} > 0 \quad on \ \mathcal{I},$$

and

$$\alpha_2 := puu^\sigma > 0 \quad on \ \mathcal{I}.$$

Proof. Assume that u is a positive solution of (2.1) on $\mathcal{I}$, and let $x \in \mathbb{D}$. Then by the Lagrange Identity (Theorem 3.4),

$$
\begin{aligned}
u(t)Lx(t) - x(t)Lu(t) &= \{u; x\}^\nabla(t) \\
Lx(t) &= \frac{1}{u(t)}\{pW(u, x)\}^\nabla(t) \\
&= \frac{1}{u(t)}\left\{puu^\sigma\left[\frac{x}{u}\right]^\Delta\right\}^\nabla(t) \\
&= \alpha_1(t)\{\alpha_2[\alpha_1 x]^\Delta\}^\nabla(t),
\end{aligned}
$$

for $t \in \mathcal{I}$, where α_1 and α_2 are as described in the theorem. □

Theorem 4.8 (Trench Factorization). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, assume $\rho(\omega) = \omega$. If (2.1) is nonoscillatory on $[a, \omega)$, then there is $t_0 \in \mathbb{T}$ such that for any $x \in \mathbb{D}$, we get the* Trench Factorization

$$Lx(t) = \beta_1(t)\{\beta_2[\beta_1 x]^\Delta\}^\nabla(t)$$

for $t \in [t_0, \omega)$, where $\beta_1, \beta_2 > 0$ on $[t_0, \omega)$, and

$$\int_{t_0}^\omega \frac{1}{\beta_2(t)} \Delta t = \infty.$$

Proof. Since (2.1) is nonoscillatory on $[a, \omega)$, (2.1) has a positive solution, u on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$. Then by Theorem 4.7, Lx has a Polya factorization on $[t_0, \omega)$. Thus there are functions α_1 and α_2 such that

$$Lx(t) = \alpha_1(t)\{\alpha_2[\alpha_1 x]^\Delta\}^\nabla(t) \text{ for } t \in [t_0, \omega),$$

defined as described in the preceding theorem. Now, if

$$\int_{t_0}^\omega \frac{1}{\alpha_2(t)} \Delta t = \infty,$$

then take $\beta_1(t) = \alpha_1(t)$, and $\beta_2(t) = \alpha_2(t)$, and we are done. Therefore, assume that

$$\int_{t_0}^\omega \frac{1}{\alpha_2(t)} \Delta t < \infty.$$

In this case, let

$$\beta_1(t) = \frac{\alpha_1(t)}{\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s} \text{ and } \beta_2(t) = \alpha_2(t) \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s \int_{\sigma(t)}^\omega \frac{1}{\alpha_2(s)}\Delta s$$

for $t \in [t_0, \omega)$. Note that as $\alpha_1, \alpha_2 > 0$, we have $\beta_1, \beta_2 > 0$ as well. Also,

$$
\begin{aligned}
\int_{t_0}^\omega \frac{1}{\beta_2(t)}\Delta t &= \lim_{b \to \omega, b \in \mathbb{T}} \int_{t_0}^b \frac{1}{\alpha_2(t) \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s \int_{\sigma(t)}^\omega \frac{1}{\alpha_2(s)}\Delta s}\Delta t \\
&= \lim_{b \to \omega, b \in \mathbb{T}} \int_{t_0}^b \left[\frac{1}{\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s} \right]^\Delta \Delta t \\
&= \lim_{b \to \omega, b \in \mathbb{T}} \left[\frac{1}{\int_b^\omega \frac{1}{\alpha_2(s)}\Delta s} \right] \\
&= \infty.
\end{aligned}
$$

Now let $x \in \mathbb{D}$. Then

$$[\beta_1 x]^\Delta(t) = \left[\frac{\alpha_1(t)x(t)}{\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s} \right]^\Delta = \frac{\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s [\alpha_1(t)x(t)]^\Delta + \alpha_1(t)x(t)\frac{1}{\alpha_2(t)}}{\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s \int_{\sigma(t)}^\omega \frac{1}{\alpha_2(s)}\Delta s}$$

for $t \in [t_0, \omega)$. So we get

$$\beta_2(t)[\beta_1(t)x]^\Delta = \alpha_2(t)[\alpha_1(t)x(t)]^\Delta \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s + \alpha_1(t)x(t)$$

for $t \in [t_0, \omega)$. Taking the ∇-derivative of both sides gives

$$
\begin{aligned}
\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla \;=\; & \left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s \\
& + \left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\rho \left[\int_t^\omega \frac{1}{\alpha_2(s)}\Delta s\right]^\nabla \\
& + [\alpha_1(t)x(t)]^\nabla
\end{aligned}
$$

for $t \in [t_0, \omega)$. We now claim that the last two terms in this expression cancel. To see this, put the expression back in terms of our positive solution u, and consider $t \in \mathbb{A}$ and $t \in \mathbb{T}_A$ separately. Careful application of Theorem 2.6 then shows that these terms do, in fact cancel, and we get

$$
\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla \int_t^\omega \frac{1}{\alpha_2(s)}\Delta s.
$$

It then follows that

$$
\beta_1(t)\left\{\beta_2(t)[\beta_1(t)x(t)]^\Delta\right\}^\nabla = \alpha_1(t)\left\{\alpha_2(t)[\alpha_1(t)x(t)]^\Delta\right\}^\nabla = Lx(t),
$$

for $t \in [t_0, \omega)$ and the proof is complete. $\qquad\square$

Theorem 4.9 (Recessive and Dominant Solutions). *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$ the we assume $\rho(\omega) = \omega$. If (2.1) is nonoscillatory on $[a, \omega)$, then there is a solution, u, called a recessive solution at ω, such that u is positive on $[t_0, \omega)$ for some $t_0 \in \mathbb{T}$, and if v is any second, linearly independent solution, called a dominant solution at ω, the following hold.*

1. $\lim_{t \to \omega^-} \frac{u(t)}{v(t)} = 0$
2. $\int_{t_0}^\omega \frac{1}{p(t)u(t)u^\sigma(t)}\Delta t = \infty$
3. $\int_b^\omega \frac{1}{p(t)v(t)v^\sigma(t)}\Delta t < \infty$ *for $b < \omega$, sufficiently close, and*
4. $\frac{p(t)v^\Delta(t)}{v(t)} > \frac{p(t)u^\Delta(t)}{u(t)}$ *for $t < \omega$, sufficiently close.*

The recessive solution, u, is unique, up to multiplication by a nonzero constant.

The proof of this theorem is directly analogous to the standard proof used in the differential equations case. See, for example, [5]

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